

# LOGIC BLOG 2013

EDITOR: ANDRÉ NIES

ABSTRACT. The 2013 logic blog has focussed on the following:

1. Higher randomness. Among others, the Borel complexity of  $\Pi_1^1$  randomness and higher weak 2 randomness is determined.
2. Reverse mathematics and its relationship to randomness. For instance, what is the strength of Jordan's theorem in analysis? (His theorem states that each function of bounded variation is the difference of two nondecreasing functions.)
3. Randomness and computable analysis. This focusses on the connection of randomness of a real  $z$  and Lebesgue density of effectively closed sets at  $z$ .
4. Exploring similarity relations for Polish metric spaces, such as isometry, or having Gromov-Hausdorff distance 0. In particular their complexity was studied.
5. Various results connecting computability theory and randomness.

Previous Logig Blogs from 2010 on can be found on Nies' web site.

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## Part 1. Higher randomness

1. GREENBERG, MONIN: AN UPPER BOUND ON THE BOREL RANK OF THE SET OF  $\Pi_1^1$ -RANDOM REALS

Written by Benoit Monin in August, joint work with Noam Greenberg.

Recall that a set  $Z \in 2^{\mathbb{N}}$  is  $\Pi_1^1$ -random if it is in no  $\Pi_1^1$  null class. Kechris [25] showed that there is a largest  $\Pi_1^1$  null class, which can be seen as a universal test for  $\Pi_1^1$ -randomness. A simple direct proof of this fact is in the last section of Hjorth and Nies [24]. For background on higher randomness see [30, Ch. 9].

We show that the set of  $\Pi_1^1$ -randoms is  $\Pi_3^0$ . Together with Yu Liang's result in Section 3, this give the exact Borel rank of the  $\Pi_1^1$ -random reals (thus also the exact Borel rank of the Kechris' largest  $\Pi_1^1$  nullset).

We will show that being  $\Pi_1^1$ -random is equivalent to a certain notion of genericity. The class of elements for this notion of genericity will have the Borel complexity of  $\Pi_3^0$ . This notion of genericity is a variation of the notion of forcing with  $\Pi_1^0$  class of positive measure, where we use the same idea lying in the difference between 1-generic and weakly-1-generic.

Following the thesis of Kautz where forcing with closed classes of positive measure is called Solovay forcing, we introduce the two notions of **weakly-Solovay- $\Sigma_1^1$ -generic** real and **Solovay- $\Sigma_1^1$ -generic** reals:

**Definition 1.1.** We say that  $X$  is weakly Solovay- $\Sigma_1^1$ -generic if for any uniformly  $\Sigma_1^1$  sequence  $\{F_n\}_{n \in \omega}$  of closed sets of positive measure with  $\lambda(\bigcup_n F_n) = 1$  we have that  $X$  is in one of the  $F_n$ .

It is easy to see that weak Solovay- $\Sigma_1^1$ -genericity is the same as higher weak 2-randomness (sometimes called strong  $\Pi_1^1$ -ML-randomness). We now give the notion of genericity that will turn out to be equivalent to  $\Pi_1^1$ -randomness:

**Definition 1.2.** We say that  $X$  is Solovay- $\Sigma_1^1$ -generic if for any uniformly  $\Sigma_1^1$  sequence  $\{F_n\}_{n \in \omega}$  of closed set of positive measure,  $\Sigma_1^1$ , we have either  $X$  is in one of the  $F_n$  or there is a  $\Sigma_1^1$  closed set of positive measure  $G$  such that  $G \cap \bigcup F_n = \emptyset$  and  $X \in G$ .

Clearly, the two last definitions are related between each other in the same way as 1-genericity is related to weak 1-genericity. This justifies the terms weakly-Solovay- $\Sigma_1^1$ -generic and Solovay- $\Sigma_1^1$ -generic. We now have to prove that this last notion of genericity coincides with  $\Pi_1^1$ -randomness. The only difficult part of the demonstration is to show that if  $X$  is Solovay- $\Sigma_1^1$ -generic then  $\omega_1^X$  is equal to  $\omega_1^{ck}$ . In order to prove this, we use the idea imagined by Sacks and simplified by Greenberg, to show that the set of  $X$  with  $\omega_1^X > \omega_1^{ck}$  has measure 0.

The idea is the following, suppose that for some  $X$  we have a function  $\varphi$  such that:

$$\forall n \ \exists \alpha < \omega_1^{ck} \ \varphi^X(n) \in \mathcal{O}_\alpha^X$$

where  $\mathcal{O}^X$  is the set of Kleene's notation for ordinals computable in  $X$  and  $\mathcal{O}_\alpha^X$  the set of Kleene's notation for ordinals computable in  $X$  with order-type strictly smaller than  $\alpha$ . Suppose also that  $X$  is Solovay- $\Sigma_1^1$ -generic.

Then we will show that the supremum of  $\varphi^X(n)$  over  $n \in \omega$  is smaller than  $\omega_1^{ck}$ . To show this we need two lemmas:

**Lemma 1.3.** *Let  $S$  be a  $\Sigma_1^1$  predicate of positive measure of the form*

$$S(X) \leftrightarrow \exists n \quad \forall \alpha < \omega_1^{ck} \quad S_{\alpha,n}(X)$$

*where  $S_{\alpha,n}$  is a  $\Delta_1^1$  predicate uniformly in  $n$  and  $\alpha$ . Then there is a union of uniformly  $\Sigma_1^1$  closed set  $\bigcup_n F_n \subseteq S$  with  $\lambda(S - \bigcup_n F_n) = 0$ .*

*Proof.* Let  $S_n = \{X \mid \forall \alpha < \omega_1^{ck} \quad S_{\alpha,n}(X)\}$ . So we have  $S = \bigcup_n S_n$  and  $S_n = \bigcap_{\alpha < \omega_1^{ck}} S_{\alpha,n}$ . Let us fix  $n$  and let us build a union of uniformly  $\Sigma_1^1$  closed set  $\bigcup_m F_{m,n} \subseteq S_n$  with  $\lambda(S_n - F_{m,n}) < 2^{-m}$ .

For each  $m$ , in each  $S_{\alpha,n}$ , find a  $\Sigma_1^1$  closed set  $F_{\alpha,m,n}$  with  $F_{\alpha,m,n} \subseteq S_{\alpha,n}$  and  $\lambda(S_{\alpha,n} - F_{\alpha,m,n}) < 2^{-p(\alpha)}2^{-m}$  where  $p$  is an injection of  $\omega_1^{ck}$  into  $\omega$ . Let us set  $F_{m,n} = \bigcap_{\alpha < \omega_1^{ck}} F_{\alpha,m,n}$ . As the intersection of closed set  $\bigcap_{\alpha} F_{\alpha,m,n}$  is a closed set and as the predicate  $\forall \alpha \quad X \in F_{\alpha,m,n}$  is clearly a  $\Sigma_1^1$  predicate, we have that  $F_{m,n}$  is a  $\Sigma_1^1$  closed set. Also as  $S_n - F_{m,n} = \bigcup_{\alpha} S_n - F_{\alpha,m,n}$  we have:

$$\begin{aligned} \lambda(S_n - F_{m,n}) &\leq \lambda(\bigcup_{\alpha} S_n - F_{\alpha,m,n}) \\ &\leq \lambda(\bigcup_{\alpha} S_{\alpha,n} - F_{\alpha,m,n}) \\ &\leq \sum_{\alpha} \lambda(S_{\alpha,n} - F_{\alpha,m,n}) \leq 2^{-m} \end{aligned}$$

Then, uniformly in  $n$  and  $m$  we have sequences of  $\Sigma_1^1$  closed set  $F_{n,m} \subseteq S_n$  such that  $\lambda(S_n - F_{n,m}) < 2^{-m}$ . Any  $\omega$ -order of  $\omega \times \omega$  gives us the desired sequence of  $\Sigma_1^1$  closed set.  $\square$

**Lemma 1.4.** *Let  $P(X)$  be a  $\Pi_1^1$  predicate of the form*

$$P(X) \leftrightarrow \forall n \quad \exists \alpha < \omega_1^{ck} \quad P_{\alpha,n}(X)$$

*where each  $P_{\alpha,n}$  is  $\Delta_1^1$  uniformly in  $n$  and  $\alpha$ . Suppose that  $X$  is Solovay  $\Sigma_1^1$ -generic and suppose  $P(X)$ . Then there exists a  $\Sigma_1^1$  closed set  $F$  of positive measure with  $X \in F$  and  $\lambda(F - P) = 0$ .*

*Proof.* If the complement of  $\{X \mid P(X)\}$  is of measure 0 then take  $F = 2^\omega$ . Otherwise from lemma 1.3 we have a union of  $\Sigma_1^1$  closed set of positive measure included in the complement and equal to it up to a set of measure 0. As  $X$  is Solovay  $\Sigma_1^1$ -generic and in  $P$  we have a  $\Sigma_1^1$  closed set of positive measure containing  $X$  which is disjoint from the complement of  $P$  up to a set of measure 0.  $\square$

We can now prove the desired theorem:

**Theorem 1.5.** *If  $Y$  is Solovay  $\Sigma_1^1$ -generic then  $\omega_1^Y = \omega_1^{ck}$ .*

*Proof.* Suppose that  $Y$  is Solovay  $\Sigma_1^1$ -generic. For any Turing functional  $\varphi^X$ , consider the set:

$$P = \{X \mid \forall n \quad \exists \alpha < \omega_1^{ck} \quad \varphi^X(n) \in \mathcal{O}_\alpha^X\}$$

Let  $P_n = \{X \mid \exists \alpha < \omega_1^{ck} \quad \varphi^X(n) \in \mathcal{O}_\alpha^X\}$  and  $P_{\alpha,n} = \{X \mid \varphi^X(n) \in \mathcal{O}_\alpha^X\}$ , so  $P = \bigcap_n P_n$  and  $P_n = \bigcup_{\alpha < \omega_1^{ck}} P_{\alpha,n}$ . Note that  $P_{\alpha,n}$  is  $\Delta_1^1$  uniformly in  $n$  and  $\alpha$ .

Suppose that  $Y$  is in  $P$ . As  $Y$  is Solovay  $\Sigma_1^1$ -generic, from the previous proposition, it is contained in a closed set of positive measure  $F$  with  $\lambda(F - P) = 0$ . In particular for each  $n$  we have  $\lambda(F - P_n) = 0$  and then  $\lambda(F^c \cup P_n) = 1$ . Then for each pair  $\langle n, m \rangle$  we can search for the smallest ordinal  $\alpha_{n,m}$  such that:

$$\lambda(F_{\alpha_{n,m}}^c \cup \bigcup_{\alpha < \alpha_{n,m}} P_{\alpha,n}) > 1 - 2^{-m}$$

where  $F_\alpha^c$  is the open set  $F^c$  enumerated up to stage  $\alpha$ . Let  $\alpha^* = \sup_{n,m} \alpha_{n,m}$ . By admissibility we have that  $\alpha^* < \omega_1^{ck}$ . Then we have:

$$\begin{aligned} \forall n \quad \lambda(F_{\alpha^*}^c \cup \bigcup_{\alpha < \alpha^*} P_{\alpha,n}) &= 1 \\ \rightarrow \forall n \quad \lambda(F_{\alpha^*} \cap \bigcap_{\alpha < \alpha^*} P_{\alpha,n}^c) &= 0 \\ \rightarrow \forall n \quad \lambda(F \cap \bigcap_{\alpha < \alpha^*} P_{\alpha,n}^c) &= 0 \\ \rightarrow \forall n \quad \lambda(F - \bigcup_{\alpha < \alpha^*} P_{\alpha,n}) &= 0 \\ \rightarrow \lambda(F - \bigcap_n \bigcup_{\alpha < \alpha^*} P_{\alpha,n}) &= 0 \end{aligned}$$

As  $X$  is Solovay- $\Sigma_1^1$  generic it is in particular weakly-Solovay- $\Sigma_1^1$  generic and then it weakly- $\Pi_1^1$ -random. In particular it belongs to no  $\Sigma_1^1$  set of measure 0. Then as  $F - \bigcap_n \bigcup_{\alpha < \alpha^*} P_{\alpha,n}$  is a  $\Sigma_1^1$  set of measure 0 we have that  $X$  belongs to  $\bigcap_n \bigcup_{\alpha < \alpha_{n,m}} P_{\alpha,n}$  and then  $\sup_n \varphi^X(n) \leq \alpha^* < \omega_1^{ck}$ .  $\square$

Using the equivalence between  $\Pi_1^1$ -random and  $\Delta_1^1$ -random +  $\omega_1^X = \omega_1^{ck}$ , we then have that the Solovay  $\Sigma_1^1$ -generic are included in the  $\Pi_1^1$ -randoms. All we have to do is prove the reverse inclusion.

**Theorem 1.6.** *The set of Solovay  $\Sigma_1^1$ -generic is exactly the set of  $\Pi_1^1$  randoms.*

*Proof.* Suppose  $X$  is not Solovay  $\Sigma_1^1$ -generic. Either  $\omega_1^X > \omega_1^{ck}$  and then  $X$  is not  $\Pi_1^1$ -random. Or  $\omega_1^X = \omega_1^{ck}$ . In this case there is a sequence of  $\Sigma_1^1$  closed set  $\bigcup_n F_n$  of positive measure such that  $X$  is not in  $\bigcup_n F_n$  and such that any  $\Sigma_1^1$  closed set of positive measure which is disjoint from  $\bigcup_n F_n$  does not contain  $X$ . The complement of  $\bigcup_n F_n$  is a  $\Pi_1^1$  set containing  $X$  that we can write as an uncountable union of Borel sets. As  $\omega_1^X = \omega_1^{ck}$  we have that  $X$  is in the first  $\omega_1^{ck}$  components of the uncountable union. Then  $X$  is in a  $\Delta_1^1$  set disjoint from  $\bigcup_n F_n$ . Since we can approximate this  $\Delta_1^1$  from below by a union of  $\Sigma_1^1$  closed set of the same measure, then  $X$  is in a  $\Pi_1^1$  set of measure 0 and then not  $\Pi_1^1$ -random.  $\square$

**Corollary 1.7.** *The set of  $\Pi_1^1$ -randoms is  $\Pi_3^0$*

The notion of test is interesting. For any union of closed  $\Sigma_1^1$  set  $S = \bigcup_n S_n$ , let us define  $\tilde{S}$  as the smallest intersection of  $\Pi_1^1$  open set  $O = \bigcap_n O_n$  containing it. Then a test is equal to  $\tilde{S} - S$ . The question of whether weakly-Solovay- $\Sigma_1^1$ -generic implies Solovay- $\Sigma_1^1$ -generic is now the same as the open question of whether weak- $\Pi_1^1$ -randomness implies  $\Pi_1^1$ -randomness. And this question is still open.

## 2. YU LIANG: A LOWER BOUND ON THE BOREL RANK OF THE SET OF $\Pi_1^1$ -RANDOM REALS

Input by Yu Liang in April.

Let  $\mathcal{S}$  be the set of  $\Pi_1^1$ -random reals. We will show that  $\mathcal{S}$  is not  $\Sigma_2^0$ . (This will be improved to not  $\Sigma_3^0$  below.) Since  $\mathcal{S}$  is a dense meager set, it is not  $\Pi_2^0$ . As noted by Chong, Nies and Yu [10, proof of Thm 3.12],  $\mathcal{S}$  is Borel:  $\mathcal{S}$  is the intersection of the  $\Delta_1^1$  randoms ( $\Pi_3^0$ ) with the sets that are low for  $\omega_1^{CK}$ , which is properly  $\Pi_{\omega_1^{CK}+2}^0$  by a result of M. Steel [37, end of Section 2].

Since  $\mathcal{S}$  is a  $\Sigma_1^1$ -set, there must be some recursive tree  $T \subseteq 2^\omega \times \omega^\omega$  such that

$$x \in \mathcal{S} \leftrightarrow \exists f \forall n (x \upharpoonright n, f \upharpoonright n) \in T.$$

Now assume for a contradiction that  $\mathcal{S}$  is a  $\Sigma_2^0$ -set. Choose a sequence of closed sets  $\{P_n\}_{n \in \omega}$  so that  $\bigcup_{n \in \omega} P_n = \mathcal{S}$ .

Recall that the Gandy topology on Cantor space is given by the  $\Sigma_1^1$  sets as a countable basis. (Note this is Polish on the set of sets that are low for  $\omega_1^{CK}$ .)

**Lemma 2.1.** *For each  $n$ , the set  $\mathcal{S} \setminus P_n$  is comeager in  $\mathcal{S}$  in the sense of the Gandy topology.*

*Proof.* Fix a  $\Sigma_1^1$  uncountable set  $C \subseteq \mathcal{S}$ . Let  $\bar{C}$  be the closure (in the Cantor space sense) of  $C$ . Then  $\bar{C}$  is a  $\Sigma_1^1$  closed set. The leftmost real in  $\bar{C}$  is either hyperarithmetic or hyperarithmetically equivalent to  $\mathcal{O}$ . So  $\bar{C}$  is not a subset of  $P_n$ . So there must be some  $\sigma \in 2^{<\omega}$  so that  $[\sigma] \cap C \neq \emptyset$  but  $[\sigma] \cap C \cap P_n = \emptyset$ . Let  $D = [\sigma] \cap C$ . So  $D \subseteq C$  is an open set (in the Gandy topology sense). Thus  $\mathcal{S} \setminus P_n$  contains a dense open set and so must be comeager in  $\mathcal{S}$ .  $\square$

Since Gandy topology has Baire property, there must be some real  $x \in \bigcap_{n \in \omega} \mathcal{S} \setminus P_n = \mathcal{S} \setminus \bigcup_{n \in \omega} P_n$ , contradiction.

### 3. YU: STRONG $\Pi_1^1$ -ML-RANDOMNESS IS PROPERLY $\Pi_3^0$

Input by Yu in May.

Strong  $\Pi_1^1$ -ML-randomness is the higher analog of weak 2-randomness. This was mentioned in a problem in [30, Ch. 9]. It is open whether this notion is the same as  $\Pi_1^1$ -randomness.

Obviously the collection of strongly  $\Pi_1^1$ -ML-random reals is  $\Pi_3^0$ .

**Proposition 3.1.** *The collection of strongly  $\Pi_1^1$ -ML-random reals is not  $\Sigma_3^0$ .*

We use a forcing argument.

Let  $\mathbb{P} = (\mathbf{P}, \leq)$  where  $\mathbf{P}$  is the collection of  $\Sigma_1^1$  closed sets with a positive measure.

Obviously, if  $g$  is sufficiently generic, then it must be strongly  $\Pi_1^1$ -ML-random.

**Lemma 3.2.** *For any  $\Sigma_1^1$  tree  $T$  with  $\mu([T]) > 0$  having only  $\Pi_1^1$ -ML random reals, there is a uniformly  $\Pi_1^1$  sequence open sets  $\{U_n\}_{n \in \omega}$  so that*

- $\forall n \mu(U_n \cap [T]) < 2^{-n}$ ; and
- for any  $\sigma$ , if  $[\sigma] \cap [T] \neq \emptyset$ , then  $[\sigma] \cap [T] \cap (\bigcap_n \{U_n\}_{n \in \omega}) \neq \emptyset$ .

*Proof.* This is like difference tests. Given  $n$ , for any  $\sigma$ , we enumerate strings of  $2^{2 \cdot |\sigma| + n + 1}$  into  $U_n$  from left to right which are possibly the leftmost finite string of length  $2 \cdot |\sigma| + n + 1$  among those in  $[\sigma] \cap [T]$ .

The sequence  $\{U_n\}_{n \in \omega}$  is precisely what we want.  $\square$

**Lemma 3.3** (Nies, see Thm. 11.7 [13]). *A  $\Pi_1^1$ -ML random real  $x$  is  $\Pi_1^1$ -difference random if and only if the  $\Pi_1^1$  version of Chaitin's halting probability  $\underline{\Omega} \not\leq_{h-T} x$ .*

**Lemma 3.4** (Greenberg, Bienvenu, Monin). *No strongly  $\Pi_1^1$ -ML random is  $h - T$ -above  $\underline{\Omega}$ . So by Lemma 3.3 every strongly  $\Pi_1^1$ -ML random real is  $\Pi_1^1$ -difference random.*

Now given ANY sequence open sets  $\{V_n\}_{n \in \omega}$ , let  $\mathcal{D}_V = \{T \mid T \in \mathbf{P} \wedge [T] \cap \bigcap_{n \in \omega} V_n = \emptyset\}$ .

**Lemma 3.5.** *If  $\bigcap_{n \in \omega} V_n$  only contains strongly  $\Pi_1^1$ -ML-random, then  $\mathcal{D}_V$  is dense.*

*Proof.* Given any condition  $T \in \mathbf{P}$ . By Lemma 3.2, there is a uniformly  $\Pi_1^1$  sequence open sets  $\{U_n\}_{n \in \omega}$  as described. Then there must be some  $\sigma$  so that  $[\sigma] \cap [T] \neq \emptyset$  but  $[\sigma] \cap [T] \cap (\bigcap_{n \in \omega} V_n) = \emptyset$  (Otherwise, there must be a real  $x \in [T] \cap (\bigcap_{n \in \omega} V_n) \cap (\bigcap_{n \in \omega} U_n)$ . Then  $x$  would be a strongly  $\Pi_1^1$ -ML random but not  $\Pi_1^1$ -difference random, a contradiction to Lemma 3.4). Let  $[S] = [\sigma] \cap [T]$ . Then  $S \leq T$  and  $S \in \mathcal{D}_V$ .  $\square$

Given any  $\Sigma_3^0$ -set, if  $g$  is sufficiently generic, then  $g$  is a strongly  $\Pi_1^1$ -ML-random real not in this set. This concludes the proof of the proposition.

#### 4. YU BASED ON GREENBERG AND MONIN: A LOWER BOUND ON THE BOREL RANK OF THE SET OF $\Pi_1^1$ -RANDOM REALS

Input by Yu in August, 2013. The result is essentially due to Greenberg and Monin. It is Greenberg who told me the result.

Let  $\mathbb{P} = (\mathbf{P}, \leq)$  where  $\mathbf{P}$  is the collection of  $\Sigma_1^1$  closed sets with a positive measure.

**Lemma 4.1.** *Every  $\mathbb{P}$ -generic real  $g$  is  $\Pi_1^1$ -random.*

*Proof.* By Theorem 1.6, it is sufficient to prove that  $g$  is  $\Sigma_1^1$ -Solovay generic.

Given a uniformly  $\Sigma_1^1$ -closed sets  $\{F_n\}_{n \in \omega}$  with positive measure. Let

$$\mathcal{D} = \{F_n \mid n \in \omega\} \cup \{F \text{ is closed and } \Sigma_1^1 \mid F \cap (\bigcup_{n \in \omega} F_n) = \emptyset\}.$$

Obviously  $\mathcal{D}$  is dense. So the lemma follows.  $\square$

Now suppose that  $\{V_n\}_n$  is a sequence open sets so that  $\bigcap_{n \in \omega} V_n$  only contains  $\Pi_1^1$  random reals. Let

$$\mathcal{D}_V = \{T \mid T \in \mathbf{P} \wedge [T] \cap \bigcap_{n \in \omega} V_n = \emptyset\}.$$

By Lemma 3.5,  $\mathcal{D}_V$  is dense.

In conclusion, the collection of  $\Pi_1^1$ -random reals is not  $\Sigma_3^0$ .

**Remark:** By the proof, for any set  $\mathcal{A}$  of reals, if  $\Pi_1^1$ -random  $\subseteq \mathcal{A} \subseteq \Pi_1^1$ -difference random, then  $\mathcal{A}$  is not  $\Sigma_3^0$ .

5. BIENVENU, GREENBERG, MONIN: A  $\Pi_1^1$ -MLR SET  $X$  IS NOT  $\Pi_1^1$ -RANDOM IFF  $X$   $hT$ -COMPUTES A  $\Pi_1^1$  SEQUENCE WHICH IS NOT  $\Delta_1^1$ .

The  $hT$ -reductions are the most general version of Turing-reductions, as defined by Bienvenu, Greenberg and Monin in LogicBlog2012. We have that if  $X$   $hT$ -computes a  $\Pi_1^1$  sequence which is not  $\Delta_1^1$ , then  $X$   $h$ -computes  $\mathcal{O}$  and is thus not  $\Pi_1^1$ -random, as  $\omega_1^X > \omega_1^{\text{CK}} \leftrightarrow X \geq_h \mathcal{O}$ . So all we need to prove is the following theorem:

**Theorem 5.1.** *If  $X$  is  $\Pi_1^1$ -MLR but not  $\Pi_1^1$ -random, then  $X$   $hT$ -computes a  $\Pi_1^1$  sequence which is not  $\Delta_1^1$ .*

*Proof.* Suppose that  $X$  is  $\Pi_1^1$ -MLR but not  $\Pi_1^1$ -random. Then from theorem 1.5 there is a uniform intersection of  $\Pi_1^1$  open sets  $\bigcap_n O_n$  so that  $X \in \bigcap_n O_n$  and so that no  $\Delta_1^1$  closed set  $F \subseteq \bigcap_n O_n$  of positive measure contains  $X$  (and thus no  $\Delta_1^1$  closed set  $F \subseteq \bigcap_n O_n$  contains  $X$ ). Let  $\{W_e\}_{e \in \omega}$  be an enumeration of the  $\Pi_1^1$  subsets of  $\omega$ . We will construct a  $\Pi_1^1$  sequence  $A$  which is not  $\Delta_1^1$  and so that  $X$  can  $hT$ -compute  $A$ . We use the usual way to make  $A$  not  $\Delta_1^1$ , by meeting each requirement

$$R_e : W_e \text{ infinite} \rightarrow A \cap W_e \neq \emptyset$$

making sure in the meantime that  $A$  is co-infinite.

In what follows, to speak of ordinal stages and finite substages in a clean way, we use the ordinal version of the euclidian division: For an ordinal  $\alpha$ , there is a unique pair of ordinal  $\langle \beta, n \rangle$  so that  $\alpha = \omega \times \beta + n$ . Furthermore one can uniformly find  $\beta$  and  $n$  from  $\alpha$  (a simple research within the ordinals smaller than  $\alpha$ ). Then, the stage  $\omega \times \beta + n$  should be understood as substage  $n$  of stage  $\alpha$ .

### Construction of $A$ :

First, for each  $e$  let  $b_e$  be a boolean initialized to 'false'. At stage  $\gamma = \omega \times \alpha + \langle e, m, k \rangle$  (At stage  $\alpha$ , at substage  $\langle e, m, k \rangle$ ), if  $b_e$  is marked 'true', go to the next stage (next substage), otherwise if  $m \in W_e[\alpha]$  with  $m > 2e$ , then consider the  $\Delta_1^1$  set  $\bigcap_n O_n[\alpha]$  and compute an increasing union of  $\Delta_1^1$  closed sets  $\bigcup_n F_n$  with  $\bigcup_n F_n \subseteq \bigcap_n O_n[\alpha]$  and  $\lambda(\bigcup_n F_n) = \lambda(\bigcap_n O_n[\alpha])$ .

If  $\lambda(F_k^c \cap O_m[\gamma]) \leq 2^{-e}$  then enumerate  $m$  into  $A$  at stage  $\gamma$ , mark  $b_e$  as 'true' and let  $U_{\langle m, e \rangle} = F_k^c \cap O_m[\gamma]$  (the  $U_{\langle m, e \rangle}$  are intended to form a higher Solovay test).

### Verification that $A$ is not $\Delta_1^1$ :

$A$  is co-infinite because for each  $e$  at most one  $m$  is enumerated into  $A$  and this  $m$  is bigger than  $2e$ . Now suppose that  $W_e$  is infinite. There exists then  $\alpha < \omega_1^{\text{CK}}$  so that  $W_e[\alpha]$  is infinite (otherwise the function which to  $n$  associates the first ordinal time at which the  $n$ -th element enters  $W_e$  would have its range cofinal in  $\omega_1^{\text{CK}}$ , which is not possible). Then there exists  $\beta > \alpha$  so that  $\lambda(\bigcap_n O_n - \bigcap_n O_n[\beta]) < 2^{-e}$ . Thus there is a  $\Delta_1^1$



closed set  $F_k \subseteq \bigcap_n O_n[\beta]$  so that  $\lambda(\bigcap_n O_n - F_k) < 2^{-e}$ . Then there exists  $a$  for that for all  $b \geq a$  we have  $\lambda(O_b - F_k) < 2^{-e}$  and in particular  $\lambda(O_b[\omega \times \beta + \omega] - F_k) < 2^{-e}$ . But as  $W_e[\alpha]$  is already infinite we have for some  $m \in W_e[\beta]$  with  $m > 2e$  that  $\lambda(O_m[\omega \times \beta + \langle e, m, k \rangle] - F_k) < 2^{-e}$  and then at stage  $\omega \times \beta + \langle e, m, k \rangle$ ,  $m$  is enumerated into  $A$ , if  $R_e$  is not met yet.

**Verification that  $\{U_{\langle m, e \rangle}\}_{m, e \in \omega}$  is a higher Solovay test:**

We put an open set in the Solovay test only when  $R_e$  is 'actively' met, and this open set has measure smaller than  $2^{-e}$ . As each  $R_e$  is 'actively' met at most once, we have a Solovay test.

**Computation of  $A$  from  $X$ :**

Note that we now just describe the algorithm to compute  $A$  from  $X$ . The verification that the algorithm works as expected is given in the next section. Let  $p$  be the smallest integer so that for any  $m > p$ ,  $X$  is in no  $U_{\langle m, e \rangle}$ . To decide whether  $m > p$  is in  $A$ , look for the smallest  $\alpha$  such that  $X \in O_m[\alpha]$ . Then decide that  $m$  is in  $A$  iff  $m$  is in  $A[\alpha]$ .

**Verification that  $X$  computes  $A$ :**

Let  $p$  be the smallest integer so that for any  $m > p$ ,  $X$  is in no  $U_{\langle m, e \rangle}$ . Suppose for  $m > p$  that  $X \in O_m[\alpha]$  and  $m \notin A[\alpha]$ . Suppose also that at latter stage  $\gamma = \omega \times \beta + \langle e, m, k \rangle > \alpha$ , the integer  $m$  is enumerated into  $A$ . By construction, it means we have  $\lambda(O_m[\gamma] - F_k) < 2^{-e}$  for some  $\Delta_1^1$  closed set  $F_k \subseteq \bigcap_n O_n$  and that  $U_{\langle m, e \rangle} = O_m[\gamma] - F_k$  (Note that  $U_{\langle m, e \rangle}$  cannot be replaced latter because of a different  $k$ , as  $R_e$  is now met).

As  $X$  does not belong to  $U_{\langle m, e \rangle}$  and does not belong to  $F_k$ , it does not belong to  $O_m[\gamma]$  which contradicts the fact that it belongs to  $O_m[\alpha] \subseteq O_m[\gamma]$ .  $\square$

## 6. BIENVENU, GREENBERG, MONIN: FOR ANY $n \geq 4$ WE HAVE $\Pi_1^1\text{-RANDOM} \leftrightarrow P_n^0(\Pi_1^1)\text{-RANDOM}$

We say that a set is  $P_2^0(\Pi_1^1)$  if it is equal to  $\bigcap_n O_n$  where each  $O_n$  is a  $\Pi_1^1$  open set uniformly in  $n$ . The  $P_3^0(\Pi_1^1)$  sets are those of the form  $\bigcap_m \bigcup_n F_{n,m}$  where each  $F_{n,m}$  is a  $\Sigma_1^1$  closed set uniformly in  $n$  and  $m$ . The  $P_n^0(\Pi_1^1)$  sets and  $S_n^0(\Pi_1^1)$  sets are then defined for any  $n \in \omega$ , following the same logic.

Let  $\mathcal{O}_1$  be a  $\Pi_1^1$  set of unique computable ordinal notations. For  $o \in \mathcal{O}_1$  we denote by  $|o|$  the corresponding ordinal. We will consider in this section an extension of the notion of functionals, which seems more adapted to work in the higher world. Some recent work of Bienvenu, Greenberg and Monin, still unpublished, says that we can have for some  $X, Y$  that  $X \geq_{hT} Y$ , but with the impossibility of having a computation of  $Y$  from  $X$  with functional consistent everywhere. This is why we decide here to make of inconsistency something 'normal' by defining  $\Pi_1^1$  "relationals" which are intended to be to

$\Pi_1^1$  relations what  $\Sigma_1^0$  functionals are to  $\Sigma_1^0$  functions.

A  $\Pi_1^1$  relational  $\varphi_P$  is given by a  $\Pi_1^1$  predicate  $P \subseteq 2^{<\omega} \times \omega \times \mathcal{O}_1$ . We write  $\varphi_P^X(n) \downarrow$  the predicate  $\exists o \exists \sigma \prec X P(\sigma, n, o)$ . If  $p \in \omega$  we write  $\varphi_P^X(n) \downarrow = p$  for  $\exists \sigma \prec X P(\sigma, n, p)$ . Note that we can have distinct values  $p_1, p_2 \in \mathcal{O}_1$  so that  $\varphi_P^X(n) \downarrow = p_1$  and  $\varphi_P^X(n) \downarrow = p_2$ . We write  $\text{dom}\varphi_P$  for the set of  $X$  such that any  $n$  is in relation with at least one element of  $p : \{X \mid \forall n \exists p \varphi_P^X(n) \downarrow = p\}$ .

**Fact 6.1.** *Each  $\Pi_1^1$  relational  $\varphi_P$  corresponds to the higher  $\Pi_2^0$  set  $\text{dom}\varphi_P$ . Conversely each higher  $\Pi_2^0$  set  $\bigcap_n O_n$  corresponds to the  $\Pi_1^1$  relational  $\varphi_P^X(n) \downarrow = p \leftrightarrow p \in \mathcal{O}_1 \wedge \exists \sigma \prec X \sigma \in O_n[[p]]$ .*

**Lemma 6.2.** *If  $\omega_1^Z > \omega_1^{\text{CK}}$  and  $Z$  is  $\Delta_1^1$  random then there is a  $\Pi_1^1$  relational  $\varphi_P$  such that  $Z \in \text{dom}\varphi_P$  and such that  $\sup_n \{\min\{|o| \mid \varphi_P^Z(n) \downarrow = o\}\} = \omega_1^{\text{CK}}$ .*

*Proof.* From theorem 1.5 there is a higher  $\Pi_2^0$  set  $\bigcap_n O_n$  containing  $Z$  so that  $Z$  is in no  $\Sigma_1^1$  closed set of positive measure included in  $\bigcap_n O_n$  (and then in no  $\Sigma_1^1$  closed set included in  $\bigcap_n O_n$ ). Consider for each  $\alpha$  computable the set  $\bigcap_n O_n[\alpha]$ . We can approximate  $\bigcap_n O_n[\alpha]$  from below by a union of  $\Delta_1^1$  closed sets of the same measure and as  $Z$  is in no  $\Delta_1^1$  nullset and in no  $\Delta_1^1$  closed sets included in  $\bigcap_n O_n[\alpha]$ ,  $Z$  cannot be in  $\bigcap_n O_n[\alpha]$ . This implies that for  $\varphi_P$  defined from  $\bigcap_n O_n$  in fact 6.1, we have  $\sup_n \{\min\{|o| \mid \varphi_P(n) \downarrow = o\}\} = \omega_1^{\text{CK}}$ .  $\square$

We now have the following lemma:

**Lemma 6.3.** *From any  $\Pi_1^1$  relational  $\varphi_P$  one can obtain effectively in  $\varepsilon$  a  $\Pi_1^1$  relational  $\varphi_Q$  so that:*

- 1 :  $\text{dom}\varphi_P = \text{dom}\varphi_Q$
- 2 :  $\forall X \forall n (\exists! o \varphi_P^X(n) = o) \rightarrow \varphi_Q^X(n) = o$
- 3 :  $\forall X \forall n \min\{|o| \mid \varphi_P^X(n) = o\} \leq \min\{|o| \mid \varphi_Q^X(n) = o\}$
- 4 :  $\lambda(\{X \mid \exists n \exists o_1 \neq o_2 \varphi_Q^X(n) \downarrow = o_1 \wedge \varphi_Q^X(n) \downarrow = o_2\}) \leq \varepsilon$

*Proof.* .

**The construction:**

In what follows, to speak of ordinal stages and finite substages in a clean way, we use the ordinal version of the euclidian division: For an ordinal  $\alpha$ , there is a unique pair of ordinal  $\langle \beta, n \rangle$  so that  $\alpha = \omega \times \beta + n$ . Furthermore one can uniformly find  $\beta$  and  $n$  from  $\alpha$  (a simple research within the ordinals smaller than  $\alpha$ ). Then, the stage  $\omega \times \beta + n$  should be understood as substage  $n$  of stage  $\alpha$ .

Take a computable sequence of rationals  $\varepsilon_n$  so that  $\sum_n \varepsilon_n \leq \varepsilon$ . For each  $n$  and uniformly in  $n$  we do the following:

At stage  $\gamma = \omega \times \alpha + \langle \sigma, o \rangle$  let  $A_\gamma = \bigcup \{[\tau] \mid \exists \beta < \gamma \ \varphi_Q^\tau(n)[\beta] \downarrow\}$ . If  $\varphi_P^\sigma(n)[\alpha] \downarrow = o$ , we effectively find a clopen set  $B_\gamma = \bigcup_{i < m} [\tau_i] \subseteq [\sigma]$  so that  $B_\gamma \cup A_\gamma$  covers  $[\sigma]$  and such that  $\lambda(B_\gamma \cap A_\gamma) < \varepsilon_n 2^{-p(\gamma)}$ , where  $p$  is an injection from  $\omega_1^{\text{CK}}$  to  $\omega$ . We then set  $\varphi_Q^{\tau_i}(n)[\gamma] = o$  for any of the  $\tau_i$  such that  $B_\gamma = \bigcup_{i < m} [\tau_i]$ .

**Verification:**

- (1) Let us prove  $\text{dom} \varphi_Q \subseteq \text{dom} \varphi_P$ . Suppose that  $\varphi_Q^X(n) \downarrow$ . Then by construction we have a stage  $\gamma$  with  $X$  in a clopen set  $B_\gamma \subseteq [\sigma]$  with  $\varphi_P^\sigma(n) \downarrow$ . Then we also have  $\varphi_P^X(n) \downarrow$  which gives us  $\text{dom} \varphi_Q \subseteq \text{dom} \varphi_P$ . For the other inclusion, suppose that  $\varphi_P^\sigma(n) \downarrow$  with  $\sigma \prec X$ . Then by construction we have a stage  $\gamma$  and an open set  $B_\gamma \cup A_\gamma$  covering  $[\sigma]$  with  $\varphi_Q^Y(n)[\gamma] \downarrow$  for any  $Y$  in  $B_\gamma \cup A_\gamma$ . Then we have that  $\varphi_Q^X(n) \downarrow$  and then  $\text{dom} \varphi_P \subseteq \text{dom} \varphi_Q$ .
- (2) Suppose that  $\exists! o \ \varphi_P^X(n) = o$ . Consider the smallest  $\gamma = \omega \times \alpha + \langle \sigma, o \rangle$  so that  $\sigma \prec X$  and  $\varphi_P^\sigma(n)[\alpha] \downarrow = o$ . By hypothesis  $\{\tau \mid \exists \beta < \gamma \ \varphi_Q^\tau(n)[\beta] \downarrow\} = \emptyset$  and then  $\varphi_Q^\sigma(n)[\gamma] \downarrow = o$ .
- (3) This is true because the images of  $n$  via  $\varphi_Q^X$  are a subset of the images of  $n$  via  $\varphi_P^X$ .
- (4) For a given  $n$  we have that  $\{X \mid \exists o_1 \neq o_2 \ \varphi_Q^X(n) \downarrow = o_1 \wedge \varphi_Q^X(n) \downarrow = o_2\}$  is included in  $\bigcup_\gamma B_\gamma \cap A_\gamma$ . Also we have  $\lambda(\bigcup_\gamma B_\gamma \cap A_\gamma) \leq \sum_\gamma \varepsilon_n 2^{-p(\gamma)} \leq \varepsilon_n$ .

□

**Lemma 6.4.** *If  $\omega_1^Z > \omega_1^{\text{CK}}$  and  $Z$  is  $\Pi_1^1$ -ML-random then there is a  $\Pi_1^1$  relational  $\varphi_P$  such that  $Z \in \text{dom} \varphi_P$ , such that  $\forall n \ \exists! o \ \varphi_P^Z(n) = o$  and such that  $\sup_n |\varphi_P(n)| = \omega_1^{\text{CK}}$ .*

*Proof.* Suppose that  $\omega_1^Z > \omega_1^{\text{CK}}$  and  $Z$  is  $\Delta_1^1$  random, from lemma 6.2 we have  $\varphi_P$  such that  $Z \in \text{dom} \varphi_P$  and such that  $\sup_n \{\min\{o \mid \varphi_P(n) \downarrow = o\}\} = \omega_1^{\text{CK}}$ . But from lemma 6.3 one can obtain uniformly in  $\varepsilon$  a functional  $\varphi_Q$  with  $\text{dom} \varphi_Q = \text{dom} \varphi_P$  and so that the  $\Pi_1^1$  open set:

$$\{X \mid \exists n \ \exists o_1 \neq o_2 \ \varphi_Q^X(n) \downarrow = o_1 \wedge \varphi_Q^X(n) \downarrow = o_2\}$$

has measure smaller than  $\varepsilon$ . Since  $Z$  is  $\Pi_1^1$ -ML random there exists a relational  $\varphi_Q$  with  $Z$  in its domain, which is functional on  $Z$  and (using 3 of lemma 6.3) such that  $\sup_n |\varphi_Q(n)| = \omega_1^{\text{CK}}$ . □

We now assume that we have  $Z$   $\Pi_1^1$ -ML-random with  $\omega_1^Z > \omega_1^{\text{CK}}$  and that we have a relational  $\varphi_P$  with the properties of lemma 6.4. In order to put  $Z$  in a  $\mathcal{P}_2^0(\Pi_1^1)$  nullset, we would like  $\text{dom} \varphi_P$  to have measure 0. In order to do so we would like  $\text{dom} \varphi_P$  to contain no  $X$  with  $\omega_1^X = \omega_1^{\text{CK}}$ . This is what we are trying to achieve now. Note that we eventually won't be able to put  $Z$  in a  $\mathcal{P}_2^0(\Pi_1^1)$  nullset, but only in a  $\mathcal{P}_4^0(\Pi_1^1)$  nullset.

For any  $e \in \omega$  we define  $R_e \subseteq \omega \times \omega$  by  $R_e(n, m) \leftrightarrow \langle n, m \rangle \in W_e$ . We define then  $R_e \upharpoonright_k$  to be  $R_e \upharpoonright_k (n, m) \leftrightarrow \langle m, k \rangle \in W_e \wedge R_e(n, m)$ . Note that

$R_e \upharpoonright_k$  is well defined for any  $e$ . Also in what follows, a morphisms from a relation  $R_a$  to another relation  $R_b$  is a function  $f$  total on  $\text{dom}R_a$ , with  $f(\text{dom}R_a) \subseteq \text{dom}R_b$  and  $R_a(x, y) \rightarrow R_b(f(x), f(y))$ . Let us consider the two following predicates on  $2^\omega \times \omega$ :

$$C_1(X, e) \leftrightarrow \exists n \exists o_n \varphi_P^X(n) \downarrow = o_n \wedge \forall f \text{ } f \text{ is not a morphism from } R_{o_n} \text{ to } R_e$$

$$C_2(X, e) \leftrightarrow \exists m \forall n \exists o_n \varphi_P^X(n) \downarrow = o_n \wedge \forall f \text{ } f \text{ is not a morphism from } R_e \upharpoonright_m \text{ to } R_{o_n}$$

We will now join them into one predicate. Let us define  $G$  to be the  $\Pi_2^0$  set of  $e$  so that  $R_e$  is a linear order of  $\omega$ . We then define:

$$C(X) \leftrightarrow X \in \text{dom}\varphi_P \wedge (\forall e \in G \ C_1(X, e) \vee C_2(X, e))$$

Let us first make sure that  $\{X \mid C(X)\}$  is  $P_4^0(\Pi_1^1)$ . We have that  $\text{dom}\varphi_P$  is  $P_2^0(\Pi_1^1)$ ,  $\{X \mid C_1(X, e)\}$  is  $S_1^0(\Pi_1^1)$  uniformly in  $e$  and the set  $\{X \mid C_2(X, e)\}$  is  $\Sigma_3^0(\Pi_1^1)$  uniformly in  $e$ . Then the set  $\text{dom}\varphi_P \cap (\{X \mid C_1(X, e)\} \cup \{X \mid C_2(X, e)\})$  is  $S_3^0(\Pi_1^1)$  uniformly in  $e$ . As  $G$  has a  $\Pi_2^0$  description, we have that  $\{X \mid C(X)\}$  is a  $P_4^0(\Pi_1^1)$  set.

The goal is now to prove that  $Z \in C$  and that for all  $X \in \text{dom}\varphi$  if  $\varphi_P$  is fonctionnal on  $X$  and  $\omega_1^X = \omega_1^{\text{CK}}$  then  $X \notin C$ .

Let us first prove that  $Z \in C$ . By hypothesis we have  $Z \in \text{dom}\varphi_P$ . Take any  $e \in G$ . Suppose that  $R_e$  is a well-founded relation. As  $e$  is already in  $G$  we have that  $R_e$  is a well-ordered relation. Then  $|R_e| < \omega_1^{\text{CK}}$ . But then there is some  $n$  so that  $|\varphi_P^Z(n)| > |R_e|$  and we cannot have a morphism from  $R_{\varphi_P^Z(n)}$  to  $R_e$ . Then  $C_1(Z, e)$  is true. Suppose now that  $R_e$  is a ill-founded relation. There is then some  $m$  so that  $R_e \upharpoonright_m$  is already ill-founded. But as  $R_{\varphi_P^Z(n)}$  is well-founded for every  $n$ , then for every  $n$  we cannot have a morphism from  $R_e \upharpoonright_m$  to  $R_{\varphi_P^Z(n)}$ . Then  $C_2(Z, e)$  is true.

Let us first prove that  $\forall X \in \text{dom}\varphi$  if  $\varphi_P$  is functional on  $X$  and  $\omega_1^X = \omega_1^{\text{CK}}$  then  $X \notin C$ . Take any  $Y \in \text{dom}\varphi_P$  so that  $\varphi_P$  is functional on  $Y$  and  $\omega_1^Y = \omega_1^{\text{CK}}$ . Then  $\sup_n |\varphi_R^Y(n)| = \alpha < \omega_1^{\text{CK}}$ . Thus there exists some code  $e \in G$  so that  $R_e$  is a well-order of order-type  $\alpha$ . For this  $e$  we certainly have for all  $n$  a morphism from  $R_{\varphi_R^Y(n)}$  into  $R_e$ . Then we do not have  $C_1(Y, e)$ . Let us now prove that we do not have  $C_2(Y, e)$ . For any  $m$  we have  $|R_e \upharpoonright_m| < \alpha$  (even if  $\alpha$  is successor). But because  $\alpha = \sup_n \varphi_R^Y(n)$  there is necessarily some  $n$  so that  $|\varphi_R^Y(n)| > |R_e \upharpoonright_m|$ . Thus there is a morphism from  $R_e \upharpoonright_m$  into  $R_{\varphi_R^Y(n)}$ . Then we do not have  $C_2(Y, e)$  and then we do not have  $C(Y)$ . Thus the measure of  $\{X \mid C(X)\}$  is bounded by the measure of

$$H = \{X \in \text{dom}\varphi_P \mid \varphi_P \text{ is not functional on } X\}$$

But we can obtain uniformly in  $\varepsilon$  some predicate  $C_\varepsilon(X)$  with  $C_\varepsilon(Z)$  and with the measure of  $H$  bounded by  $\varepsilon$ . As each  $C_\varepsilon$  is  $P_4^0(\Pi_1^1)$  uniformly in  $\varepsilon$  we have that  $\bigcap_\varepsilon C_\varepsilon$  is a  $P_4^0(\Pi_1^1)$  set of measure 0 and containing  $Z$ . Thus  $P_4^0(\Pi_1^1)$  nullsets are enough to capture anything that a  $\Pi_1^1$  nullset can capture. In particular the  $P_n^0(\Pi_1^1)$  sets do not capture anything more for  $n > 4$ .

## Part 2. Reverse Mathematics

### 7. NIES: THE STRENGTH OF JORDAN DECOMPOSITION FOR FUNCTIONS OF BOUNDED VARIATION

Written by Nies based on work with N. Greenberg, J.S. Miller, and T. Slaman.

All real valued functions will have domain  $[0, 1]$  unless otherwise mentioned. Variables  $f, g, h$  denote functions. Recall that for a function  $f$ , one defines the variation function  $V_f$  by

$$V_f(x) = \sup \sum_{i=1}^{n-1} |f(t_{i+1}) - f(t_i)| < \infty,$$

where the sup is taken over all collections  $t_1 \leq t_2 \leq \dots \leq t_n$  in  $[0, x]$ . One says that  $f$  is of bounded variation if  $V_f(1) < \infty$ .

The Jordan decomposition theorem states that every function  $f$  of bounded variation can be written in the form  $f = g - h$  where  $g, h$  are nondecreasing. Moreover, if  $f$  is continuous, we can ensure that  $g, h$  are continuous as well. This is easily proved: let  $g = V_f$ , and observe that  $h = g - f$  is nondecreasing. So we have a Jordan decomposition  $f = g - (g - f)$ . Jordan proved this theorem in his lectures at the Ecole Polytechnique 1882-7. Today it is often treated in a first course on real analysis. Its strength in the sense of reverse mathematics is not obvious. We will see that, depending on whether we want  $g, h$  to be continuous or not, gives rise to principle equivalent to  $\text{ACA}_0$ , or to  $\text{WKL}_0$ .

Let us say that

$$f \leq_{\text{slope}} g :\Leftrightarrow \forall x < y [f(y) - f(x) \leq g(y) - g(x)].$$

That is, the slopes of  $g$  are at least as large as the slopes of  $f$ . Clearly, this is equivalent to saying that  $h := g - f$  is nondecreasing. Thus, the problem of finding a Jordan decomposition of  $f$  is equivalent to finding a nondecreasing function  $g$  with  $f \leq_{\text{slope}} g$ . This was already pointed out in [41]. (Sometimes one of the functions is partial; then we only look at slopes in the domain.)

**7.1. Jordan decomposition via continuous functions.** We first work in the usual setting of reverse mathematics, which only deals with continuous functions, suitably encoded, for instance, by the values at rationals, together with a modulus of uniform continuity (see e.g. Simpson's book II.6). It is equivalent (over  $\text{RCA}_0$ ) to describe  $f$  as the limit of a Cauchy name  $\langle p_s \rangle$  with respect to the sup norm, where the  $p_s$  are polygonal functions with rational breakpoints. Thus  $\|p_t - p_s\|_{\text{sup}} \leq 2^{-s}$  for  $t > s$ .

The principle  $\text{Jordan}_{\text{cont}}$  says that for each (continuous) function  $f$  of bounded variation, there are continuous nondecreasing functions  $g, h$  such that  $f = g - h$ .

In Proposition 7.1 and Theorem 7.4 below, technique and some writing involving reverse mathematics was provided by Keita Yokoyama.

**Proposition 7.1.** *Over  $\text{RCA}_0$ , we have*

$$\text{Jordan}_{\text{cont}} \leftrightarrow \text{ACA}_0.$$

*Proof.*  $\leftarrow$ : given  $f$ , from the jump of a representation of  $f$  as a continuous function we can compute a representation of  $V_f$  as a continuous function. This works in  $\mathbf{ACA}_0$ .

$\rightarrow$ : Suppose we are given a model of  $\mathbf{Jordan}_{\text{cont}}$ . Let

$$q_n = 1 - 2^{-n}, \text{ and } q_{n,s} = q_n - 2^{-n-s-1}.$$

Instead of proving  $\mathbf{ACA}_0$ , we will show the existence of the range of any one-to-one functions on  $\mathbb{N}$  within  $\mathbf{RCA}_0$ . (See [35, Lemma III.1.3]).

Let  $h : \mathbb{N} \rightarrow \mathbb{N}$  be a one-to-one function. Define continuous functions  $f_s : [0, 1] \rightarrow \mathbb{R}$  as follows: define  $f_s$  on  $[q_{n,k}, q_{n,k+1}]$  to be a sawtooth function of height  $2^{-k}$  with  $2^{k-n}$  many teeth if  $k \leq s$  and  $h(k) = n$ , and  $f_s = 0$  otherwise. Then, the limit  $f = \lim_{s \rightarrow \infty} f_s$  exists.

Let  $f \leq_{\text{slope}} g$ . Take a function  $\eta : \mathbb{N} \rightarrow \mathbb{N}$  so that  $g(q_n) - g(q_{n,\eta(n)}) < 2^{-n}$ . This can be done by the following argument: since  $g$  is continuous and  $\lim_{k \rightarrow \infty} q_{n,s} = q_n$ , we have

$$\forall n \exists k \theta(n, k) \equiv \exists m \theta_0(n, m, k)$$

where  $\theta(n, k)$  is a  $\Sigma_1^0$ -formula which expresses  $g(q_n) - g(q_{n,k}) < 2^{-n}$ , and  $\theta_0(n, m, k)$  is a  $\Sigma_0^0$ -formula. Define  $\eta_0$  as  $\eta_0(n)$  to be the least  $\langle m, k \rangle$  where  $\langle \cdot, \cdot \rangle$  is a standard pairing function, and  $\eta(n) = (\eta_0(n))_1$ .

Now, if  $h(k) = n$ , then  $g(q_n) - g(q_{n,k}) \geq g(q_{n,k+1}) - g(q_{n,k}) \geq 2^{-n}$ , and hence  $k < \eta(n)$ . Thus,  $n \in \text{rng}(h) \Leftrightarrow \exists k < \eta(n) \ h(k) = n$ , which means that the range of  $h$  exists by  $\Delta_1^0$ -comprehension.  $\square$

**Corollary 7.2.** *Over  $\mathbf{RCA}_0$ , the statement that every (continuous) function  $f$  of bounded variation has a variation function is equivalent to  $\mathbf{ACA}_0$ .*

*Proof.* Given  $X$ , let  $f$  be the function constructed above. If  $V_f$  exists then  $V_f - (V_f - f)$  is a continuous Jordan decomposition. So  $V_f$  computes  $X'$ .  $\square$

**7.2. Nies, Yokoyama: Jordan decomposition without continuity.** A weaker principle is obtained if we admit non continuity of  $g, h$  in a Jordan decomposition  $f = g - h$ . We only require that  $g, h$  are defined in  $I_{\mathbb{Q}} := \mathbb{Q} \cap [0, 1]$ . An  $\mathbb{R}$ -valued function  $g$  defined on  $I_{\mathbb{Q}}$  is given by a path  $Z_f$  through a binary tree. Let  $\langle p_n, q_n \rangle$  be a list of all pairs of rationals  $\langle p, q \rangle$  with  $0 \leq p \leq 1$ . We let  $Z_f(2n) = 1$  iff  $g(p_n) < q_n$ . We let  $Z_f(2n+1) = 1$  iff  $g(p_n) > q_n$ . We often identify  $f$  and  $Z_f$ . It is clear that the nondecreasing functions form a  $\Pi_1^0$  class.

The principle  $\mathbf{Jordan}_{\mathbb{Q}}$  says that for each (necessarily continuous by the encoding) function  $f$  of bounded variation, there are nondecreasing functions  $g, h$  defined on  $I_{\mathbb{Q}}$  such that  $f = g - h$  on  $I_{\mathbb{Q}}$ . Letting  $\hat{g}(x) = \sup\{g(q) \mid q \leq x \wedge q \in I_{\mathbb{Q}}\}$ , we obtain an actual Jordan decomposition because  $f \leq_{\text{slope}} \hat{g}$ .

We first prove a purely computability theoretic result. Yokoyama has given the extension to reverse mathematics- see below.

**Theorem 7.3** (Greenberg, Miller, Nies, Slaman, 2013). *An oracle  $B$  is PA-complete  $\Leftrightarrow$  for each computable function  $f$  on  $[0, 1]$  of bounded variation,  $B$  computes a function  $g : I_{\mathbb{Q}} \rightarrow \mathbb{R}$  with  $f \leq_{\text{slope}} g$ .*

*Proof.*  $\leftarrow$ : Given  $f$ , via the encoding above, the functions  $g$  defined on  $I_{\mathbb{Q}}$  with  $f \leq_{\text{slope}} g$  form a nonempty  $\Pi_1^0(f)$  class. Then  $B$  computes a member of this class.

$\rightarrow$ : We define a computable function  $f$  of bounded variation on  $[0, 1]$  such that each function  $g: I_{\mathbb{Q}} \rightarrow \mathbb{R}$  with  $f \leq_{\text{slope}} g$  has PA degree.

Let  $\mathcal{P} \subseteq 2^{\mathbb{N}}$  be a nonempty  $\Pi_1^0$  class of sets of PA degree, such as the (binary encoded) completions of Peano arithmetic. As usual  $\mathcal{P}_s$  is a clopen set computable from  $s$  approximating  $\mathcal{P}$  at stage  $s$ . So  $\mathcal{P} = \bigcap_s \mathcal{P}_s$ . By standard methods there is a computable prefix-free sequence  $\langle \sigma_s \rangle_{s \in \mathbb{N}}$  of strings of length  $s$  such that  $[\sigma_s] \cap \mathcal{P}_s \neq \emptyset$  for each  $n$ .

Given  $\sigma \in 2^{<\omega}$ , let  $I_{\sigma} = [0.\sigma, 0.\sigma + 2^{-|\sigma|}]$  be the corresponding closed subinterval of  $[0, 1]$ . By stage  $s$  we determine  $f$  up to a precision of  $2^{-s}$ . Suppose  $n$  enters  $\emptyset'$  at stage  $s$ . Let  $\sigma = \sigma_s$ . We define  $f$  on  $I_{\sigma}$  to be a sawtooth function of height  $2^{-s}$  with  $2^{s-n}$  many teeth. It is clear that this adds at most  $2^{-n+1}$  to the variation of  $f$ . So  $f$  is of bounded variation. (It is in fact AC since it can be written as an integral.)

Now suppose  $g: I_{\mathbb{Q}} \rightarrow \mathbb{R}$  is a function such that  $f \leq_{\text{slope}} g$ . As before for  $x \in [0, 1]$  let  $\hat{g}(x) = \sup\{g(q) \mid q \leq x \wedge q \in I_{\mathbb{Q}}\}$ .

*Case 1.*  $\hat{g}$  is discontinuous at the real  $y = 0.Y$  for some  $Y \in \mathcal{P}$ . Then  $Y \leq_T g$ , so  $g$  is of PA degree. (To see this, fix rational  $r$  with  $\hat{g}(y) < r < g^+(y)$ . Then  $p < y \leftrightarrow g(p) < r$ , and  $p > y \leftrightarrow g(p) > r$ .)

*Case 2.* Otherwise. Then  $\emptyset' \leq_T g$ : given  $n$ , using  $g$  compute stage  $s$  such that for each  $\sigma$  of length  $s$  with  $[\sigma] \cap \mathcal{P}_s \neq \emptyset$ , we have  $g(\max I_{\sigma}) - g(\min I_{\sigma}) < 2^{-n}$ . This  $s$  exists by case assumption using compactness of Cantor space. (This part of the argument cannot be adapted to reverse mathematics.) Then as before we have  $n \in \emptyset' \leftrightarrow n \in \emptyset'_s$ .  $\square$

Yokoyama, starting from the proof of Theorem 7.3 above, has provided a proof that works in reverse mathematics.

**Theorem 7.4.** *Over  $\text{RCA}_0$ , we have*

$$\text{Jordan}_{\mathbb{Q}} \leftrightarrow \text{WKL}_0.$$

*Proof of Theorem 7.4.*  $\leftarrow$ : the original proof can be carried out within  $\text{WKL}_0$ .

$\rightarrow$ : we reason within  $\text{RCA}_0$ . Let  $T \subseteq 2^{<\mathbb{N}}$  be an infinite binary tree. We will show that  $T$  has a path. Let  $\tilde{T} = \{\tau \notin T \mid \tau \upharpoonright (|\tau| - 1) \in T\}$ . Without loss of generality, we may assume that  $\tilde{T}$  is infinite. Define  $h: \mathbb{N} \rightarrow 2^{<\mathbb{N}}$  as  $h(n)$  to be (one of) the shortest leaf (dead end) of  $T \setminus \{h(k) \mid k < n\}$ . (Note that  $h$  can be considered as  $h: \mathbb{N} \rightarrow \mathbb{N}$  by the usual coding.) Then, we can easily see that  $T \setminus \text{rng}(h) = \text{Ext}(T) = \{\sigma \in T \mid \sigma \text{ has infinitely many extensions in } T\}$ . Let  $\langle \tilde{\sigma}_k \mid k \in \mathbb{N} \rangle$  be an enumeration of  $\tilde{T}$  such that  $|\tilde{\sigma}_i| \leq |\tilde{\sigma}_{i+1}|$ . By an easy calculation,  $|\tilde{\sigma}_k| \leq l$  implies  $k \leq 2^l$ . For given  $\sigma \in 2^{<\mathbb{N}}$ , define  $I_{\sigma} = [l_{\sigma}, r_{\sigma}] = [0.\sigma, 0.\sigma + 2^{-|\sigma|}]$ . Now, define continuous functions  $f_s: [0, 1] \rightarrow \mathbb{R}$  as follows: define  $f_s$  on  $I_{\tilde{\sigma}_k}$  to be a sawtooth function of height  $2^{-k}$  with  $2^{k-n}$  many teeth if  $k \leq s$  and  $h(k) = n$ , and  $f_s = 0$  otherwise. Then, the limit  $f = \lim_{s \rightarrow \infty} f_s$  exists.

Now suppose  $g: I_{\mathbb{Q}} \rightarrow \mathbb{R}$  is a function such that  $f \leq_{\text{slope}} g$ . Define  $\Delta: \mathbb{N} \rightarrow \mathbb{R}$  as

$$\Delta(k) = \max\{g(r_{\sigma}) - g(l_{\sigma}) \mid \sigma \in T \wedge |\sigma| = k\}.$$

Note that  $\Delta(k) < 2^{-n}$  can be expressed by a  $\Sigma_1^0$ -formula.

*Case 1:*  $\lim_{n \rightarrow \infty} \Delta(n) = 0$ . In this case, using the same argument as the above proof, take  $\eta : \mathbb{N} \rightarrow \mathbb{N}$  so that  $\Delta(\eta(n)) < 2^{-n}$ . If  $h(k) = n$ , then,  $g(r_{\tilde{\sigma}_k}) - g(l_{\tilde{\sigma}_k}) \geq 2^{-n}$ , hence,  $|\tilde{\sigma}_k| \leq \eta(n)$ . Thus,  $n \in \text{rng}(h) \Leftrightarrow \exists k \leq 2^{\eta(n)} h(k) = n$ , thus,  $T \setminus \text{rng}(h) = \text{Ext}(T)$  exists. Hence, we can easily find a path of  $T$ .

*Case 2:*  $\lim_{n \rightarrow \infty} \Delta(n) > 0$ . In this case, take  $q \in \mathbb{Q}$  such that  $\lim_{n \rightarrow \infty} \Delta(n) > q > 0$ , and define  $\hat{T}$  as  $\hat{T} = \{\sigma \in T \mid g(r_\sigma) - g(l_\sigma) > q\}$ . Then,  $\hat{T}$  is an infinite subtree of  $T$ , and there exists  $K \in \mathbb{N}$  such that for any prefix-free  $P \subseteq \hat{T}$ ,  $|P| \leq K$ . For this, take  $K$  so that  $Kq > g(1) - g(0)$ . Then, for any prefix-free  $P \subseteq \hat{T}$ ,

$$|P|q < \sum_{\sigma \in P} g(r_\sigma) - g(l_\sigma) \leq g(1) - g(0) < Kq.$$

Thus, we have  $|P| \leq K$ . Now, we can find a path of  $\hat{T}$  by the following claim.

*Claim (RCA<sub>0</sub>).* If  $T$  is an infinite binary tree, and there exists  $K \in \mathbb{N}$  such that for any prefix-free  $P \subseteq T$ ,  $|P| \leq K$ , then  $T$  has a path.

By  $\Sigma_1^0$ -induction, take

$$k = \max\{i \leq K \mid \exists P \subseteq T \text{ such that } P \text{ is prefix-free and } |P| = i\},$$

and let  $P_k \subseteq T$  be its witness. Then, any long enough  $\sigma \in T$  is an extension of a member of  $P_k$ , and it has at most one successor. Thus, by  $\Sigma_1^0$ -induction, there exists  $\tau \in P_k$  such that  $\tau \in \text{Ext}(T)$ . Since each extension of  $\tau$  has exactly one successor, we can easily find a path of  $T$  extending  $\tau$ .  $\square$

A stronger variant **strong Jordan<sub>Q</sub>** would require that  $f$  is only defined on  $I_{\mathbb{Q}}$ , and of bounded variation for partitions consisting of rationals. By the proof above, this principle is also equivalent to  $\text{WKL}_0$  over  $\text{RCA}_0$ .

## 8. YOKOYAMA : NOTES ON BVDIFF AND REVERSE MATHEMATICS

The following was contributed by Keita Yokoyama<sup>1</sup> following a talk of Nies at JAIST (Kanazawa, Japan) suggesting the principles studied below. The principle BVDiff was introduced by Greenberg, Nies and Slaman in Auckland, Nov 2012.

A function  $f : \subseteq [0, 1] \rightarrow \mathbb{R}$  with domain containing  $\mathbb{Q} \cap [0, 1]$  is said to be pseudo-differentiable at  $z \in (0, 1)$  if  $f(z) := \lim_{x \in \mathbb{Q}, x \rightarrow z} f(x)$  exists and for every  $\varepsilon > 0$ , there exists  $\delta > 0$  such that for every  $0 < |h|, |h'| < \delta$ ,

$$\left| \frac{f(z+h) - f(z)}{h} - \frac{f(z+h') - f(z)}{h'} \right| < \varepsilon.$$

Note that this is equivalent (over  $\text{RCA}_0$ ) to the definition of pseudo-differentiability in [7, Section 7]. In the reverse mathematics setting, note that we don't require that the derivative exists as a real of the model.

**Theorem 8.1.** *The following are equivalent over  $\text{RCA}_0$ .*

- (1)  $\text{WWKL}_0$ .

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<sup>1</sup> School of Information Science, Japan Advanced Institute of Science and Technology, 1-1 Asahidai, Nomi, Ishikawa, 923-1292, JAPAN, E-mail: y-keita@jaist.ac.jp



- (2) BVDiff: every continuous bounded variation function  $f : [0, 1] \rightarrow \mathbb{R}$  has a pseudo-differentiable point.
- (3) aeBVDiff: every continuous bounded variation function  $f : [0, 1] \rightarrow \mathbb{R}$  is pseudo-differentiable almost surely in the following sense:  
for any family of open intervals  $\mathcal{U} = \{(u_i, v_i)\}_{i \in \mathbb{N}}$ , if  $\mathcal{U}$  covers any pseudo-differential points of  $f$ , then  $\sum_{i \in \mathbb{N}} (v_i - u_i) \geq 1$ .

This theorem follows from the Greenberg/Miller/Nies/Slaman results in Section 7, Brattka/Miller/Nies [7], and a new fact:

**Proposition 8.2.** BVDiff (or aeBVDiff) is provable within  $\text{WWKL}_0$ .

**Lemma 8.3** (Brattka/Miller/Nies [7],  $\text{RCA}_0$ ). Let  $f : \mathbb{Q} \cap [0, 1] \rightarrow \mathbb{R}$  be a non-decreasing function, and let  $z \in [0, 1]$  be Martin-Löf (computably) random relative to  $f$ . Then,  $f$  is pseudo-differentiable at  $z$ .

Recall Theorem 7.4 above: The following are equivalent over  $\text{RCA}_0$ .

- (1)  $\text{WKL}_0$ .
- (2) For every bounded variation function  $f : \mathbb{Q} \cap [0, 1] \rightarrow \mathbb{R}$ , there exist non-decreasing functions  $g, h : \mathbb{Q} \cap [0, 1] \rightarrow \mathbb{R}$  such that  $f = g - h$ .

Another crucial ingredient is the following, due to Stephen G. Simpson and Keita Yokoyama.

**Lemma 8.4** ([36], Lemma 3.6). For any countable model  $(M, S) \models \text{WWKL}_0$ , there exists  $\bar{S} \supseteq S$  such that  $(M, \bar{S}) \models \text{WKL}_0$  and the following holds:

- (†) for any  $A \in \bar{S}$  there exists  $B \in S$  such that  $B$  is Martin-Löf random relative to  $A$ .

*Proof of Proposition 8.2.* We will show that BVDiff holds in any countable model of  $\text{WWKL}_0$ . Let  $(M, S)$  be a countable model of  $\text{WWKL}_0$ , and let  $f : [0, 1] \rightarrow \mathbb{R}$  be a continuous bounded variation function in  $(M, S)$ . By Lemma 8.4, take an extension  $\bar{S} \supseteq S$  such that  $(M, \bar{S}) \models \text{WKL}_0$  and satisfies (†). Then, by Theorem 7.4, there exist non-decreasing functions  $g, h \in \bar{S}$ ,  $g, h : \mathbb{Q} \cap [0, 1] \rightarrow \mathbb{R}$  such that  $f = g - h$ . By (†), take  $z \in S$ ,  $z \in [0, 1]$  which is Martin-Löf random relative to  $g \oplus h$  (in  $(M, \bar{S})$ ). Then, by Lemma 8.3, both of  $g$  and  $h$  are pseudo-differentiable at  $z$  (in  $(M, \bar{S})$ ), thus,  $f$  is pseudo-differentiable at  $z$  (in  $(M, S)$ ).

To show aeBVDiff, let  $\mathcal{U} = \{(u_i, v_i)\}_{i \in \mathbb{N}}$  be a family of open intervals such that  $\sum_{i \in \mathbb{N}} (v_i - u_i) < 1$ . Then,  $[0, 1] \setminus \bigcup_{i \in \mathbb{N}} (u_i, v_i)$  is a closed set which has a positive measure. Thus, in the above proof, we can find a Martin-Löf random point  $z$  in  $[0, 1] \setminus \bigcup_{i \in \mathbb{N}} (u_i, v_i)$ .  $\square$

**Remark 8.5.** Within  $\text{WWKL}_0$  one cannot assure the existence of the value of the derivative, in other words, the pseudo-differentiability in BVDiff cannot be replaced with the (usual) differentiability. In fact, Jason Rute showed that the following are equivalent over  $\text{RCA}_0$ .

- (1) Every continuous bounded variation function  $f : [0, 1] \rightarrow \mathbb{R}$  has a derivative somewhere, i.e., there exists  $z \in (0, 1)$  such that  $f'(z) = \lim_{x \rightarrow z} (f(x) - f(z))/(x - z)$  exists.
- (2)  $\text{ACA}_0$ .

## 9. RANDOMNESS NOTIONS AS PRINCIPLES OF REVERSE MATHEMATICS

Written by Nies based on work with Greenberg and Slaman in Cambridge, June 2012 and Auckland, Dec 2012.

Let  $C$  denote a randomness notion. For instance  $MLR$  is ML-randomness,  $CR$  is computable randomness and  $SR$  is Schnorr randomness. We study the strength of the system

$$C_0 = RCA_0 + \forall X \exists Y [Y \in C^X].$$

Note that  $MLR_0$  is equivalent to weak weak König's lemma at least for  $\omega$ -models.

**Proposition 9.1.**  *$CR_0$  does not imply  $MLR_0$ , as shown by a suitable  $\omega$ -model.*

This suggests to call the principle  $CR_0$  weak weak weak König's lemma. Recall that every high set is Turing above a computably random set by [32] (also see [30, Ch. 7]).

*Proof.* By the proof of [9, Lemma 4.11], for each set  $B$  of non-d.n.c. degree there is a set  $X$  high (even LR-hard) relative to  $B$  such that  $B \oplus X$  is also not of d.n.c. degree. Iterating this in the standard way, we build an  $\omega$ -model of  $CR_0$  without a set of d.n.c. degree. In particular, there is no ML-random set.  $\square$

Recall from [32] that every non high Schnorr random set is already ML-random. This only requires  $\Sigma_1$ -induction. The following is somewhat surprising and maybe explains why these two randomness notions are harder to separate than other pairs of notions.

**Proposition 9.2.**  *$CR_0$  is equivalent to  $SR_0$ .*

*Proof.* Let  $\mathcal{M}$  be a model of  $SR_0$ . Let  $X$  be a set of  $\mathcal{M}$ . Arguing within  $\mathcal{M}$ , if no set  $Y$  is high in  $X$ , then  $SR^X = MLR^X$ , so by assumption on  $\mathcal{M}$  there is a  $Z$  in  $CR^X$ . Otherwise, some set  $Y$  is high in  $X$ , i.e.,  $X'' \leq_T (Y \oplus X)'$ , and so  $Y \oplus X$  computes a set in  $CR^X$ .  $\square$

We note that analytical equivalents such as  $BVDiff$  of a randomness axiom are harder to come by in the absence of a universal test. However, we note that  $CR_0$  is equivalent, over  $RCA_0 +$  sufficient induction, to the statement that for each  $X$  there is real  $z$  such that each nondecreasing function  $f$  computable from  $X$  is differentiable at  $z$ .

**Question 9.3.** Find other pairs of randomness notions close to each other where the corresponding principles are equivalent. For instance, consider pairs among  $MLR_0$ , difference random reals, ML-random density one points, Oberwolfach random reals.

$MLR_0$  and  $DiffR_0$  should be equivalent over  $RCA_0 +$  by an argument similar to 9.2.

### Part 3. Randomness and computable analysis

#### 10. NIES: DENSITY AND DIFFERENTIABILITY: DYADIC VERSUS FULL

For  $U, V$  measurable subsets of a measure space  $(X, \mu)$ , if  $\mu(V) > 0$  we let

$$\mu_V(U) = \mu(U \cap V) / \mu(V).$$

This is the local, or conditional, measure of  $U$  with respect to  $V$ .

The definitions below follow [5]. Let  $\lambda$  denote Lebesgue measure.

**Definition 10.1.** We define the (lower Lebesgue) density of a set  $\mathcal{C} \subseteq \mathbb{R}$  at a point  $z$  to be the quantity

$$\varrho(\mathcal{C}|z) := \liminf_{\gamma, \delta \rightarrow 0^+} \frac{\lambda([z - \gamma, z + \delta] \cap \mathcal{C})}{\gamma + \delta}.$$

(If we let  $I = [z - \gamma, z + \delta]$ , the expression above turns into the local measure  $\lambda_I(\mathcal{C})$ .)

Intuitively, this measures the fraction of space filled by  $\mathcal{C}$  around  $z$  if we “zoom in” arbitrarily close. Note that  $0 \leq \varrho(\mathcal{C}|z) \leq 1$ . We will first discuss the Lebesgue density theorem.

**Theorem 10.2** (Lebesgue density theorem). *Let  $\mathcal{C} \subseteq \mathbb{R}$  be a measurable set. Then  $\varrho(\mathcal{C}|z) = 1$  for almost every  $z \in \mathcal{C}$ .*

It is interesting to compare this modern formulation with the original in [26, page 407]:

Raisonnant de même sur la densité à gauche, on voit finalement que la densité d'un ensemble mesurable est égale à un en presque tous les points de cet ensemble.

In 1910, mathematical writing was rather different from what it is today. The statement above is at the *end* of a long argument. None of the statements in the 90-page monograph is labelled; so there is no cross referencing.

A (closed) basic dyadic interval has the form  $[r2^{-n}, (r+1)2^{-n}]$  where  $r \in \mathbb{Z}, n \in \mathbb{N}$ . The lower dyadic density of a set  $\mathcal{C} \subseteq \mathbb{R}$  at a point  $z$  is the variant one obtains when only considering basic dyadic intervals containing  $z$ :

$$\varrho_2(\mathcal{C}|z) := \liminf_{z \in I \wedge |I| \rightarrow 0} \frac{\lambda(I \cap \mathcal{C})}{|I|},$$

where  $I$  ranges over basic dyadic intervals containing  $z$ . Clearly  $\varrho_2(\mathcal{C}|z) \geq \varrho(\mathcal{C}|z)$ . Sometimes we use *open* basic dyadic intervals; for the definition above the distinction does not matter.

Suppose that a real  $z$  is not a dyadic rational. Let  $0.Z$  be its binary expansion. Note that  $\varrho_2(\mathcal{C}|z)$  is the same as

$$\liminf_{\sigma \prec Z \wedge |\sigma| \rightarrow \infty} \frac{\lambda([\sigma] \cap \mathcal{C})}{2^{-|\sigma|}},$$

when we view  $\mathcal{C}$  as a subclass of  $2^{\mathbb{N}}$ . This is the density in Cantor space.

**Definition 10.3** ([5]). Consider  $z \in [0, 1]$ .

- We say that  $z$  is a *density-one point* if  $\varrho(\mathcal{C}|z) = 1$  for every effectively closed class  $\mathcal{C}$  containing  $z$ .
- We say that  $z$  is a *positive density point* if  $\varrho(\mathcal{C}|z) > 0$  for every effectively closed class  $\mathcal{C}$  containing  $z$ .

By the Lebesgue density theorem and the fact that there are only countably many effectively closed classes, almost every real  $z$  is a density-one point. Note that we can form similar definitions with dyadic density. The distinction between positive and full density is typical for the setting of effective analysis. In classical analysis, everything is settled by Lebesgue's theorem. In effective analysis, more randomness is required to ensure a real is a full density-one point. Day and Miller [11] have built a ML-random real which is a positive density point but not density-one point. They can in fact ensure this real is  $\Delta_2^0$ .

A closely related notion, *non-porosity*, originates in the work of Denjoy. See for instance [6, 5.8.124] (but note the typo in the definition there). We say that a set  $\mathcal{C} \subseteq \mathbb{R}$  is *porous at  $z$*  via the constant  $\varepsilon > 0$  if there exists arbitrarily small  $\beta > 0$  such that  $(z - \beta, z + \beta)$  contains an open interval of length  $\varepsilon\beta$  that is disjoint from  $\mathcal{C}$ . We say that  $\mathcal{C}$  is *porous at  $z$*  if it is porous at  $z$  via some  $\varepsilon > 0$ .

**Definition 10.4** ([5]). We call  $z$  a *porosity point* if some effectively closed class to which it belongs is porous at  $z$ . Otherwise,  $z$  is a *non-porosity point*.

Clearly, if  $\mathcal{C}$  is porous at  $z$  then  $\varrho(\mathcal{C}|z) < 1$ , so  $z$  is not a density-one point. Therefore, almost every point of  $\mathcal{C}$  is a non-porosity point.

**10.1. Dyadic density 1 is equivalent to full density 1 for ML-randoms reals.** [5, Remark 3.4] show that a ML-random real which is a *dyadic* positive density point already is a full positive density point; both notions coincide with difference randomness, or equivalently, being ML-random and Turing incomplete [16]. Mushfeq Khan and Joseph Miller have recently proved the analog of this result for density-one points.

**Theorem 10.5.** *Let  $z$  be a ML-random dyadic density-one point. Then  $z$  is a full density-one point.*

The actual statement Joe and Mushfeq proved is the following.

**Theorem 10.6.** *Suppose  $z$  is a non-porosity point. Let  $\mathcal{P}$  be a  $\Pi_1^0$  class,  $z \in \mathcal{P}$ , and  $\rho(\mathcal{P} | z) < 1$ . Then already  $\rho_2(P | z) < 1$ .*

Thus, we have the same class  $\mathcal{P}$  both times. This implies Theorem 10.5 since  $z$  is Turing incomplete, and hence by the result of Bienvenu et al. [5] a non-porosity point. In Remark 10.17 we will see that in fact  $\rho(\mathcal{P} | z) = \rho_2(P | z)$  for each non-porosity point  $z$ .

*Proof of Thm. 10.6.* Let  $\epsilon > 0$  be such that  $\rho(P | z) < 1 - \epsilon$ . Assume that  $\rho_2(P | z) = 1$ . Let  $n^*$  be sufficiently large so that  $\lambda_L(\mathcal{P}) \geq 1 - \epsilon/4$  for each basic dyadic interval of length  $\leq 2^{-n^*}$  containing  $z$ .

Consider now an arbitrary interval  $I$  of length  $\leq 2^{-n^*}$  with  $z \in I$  and  $\lambda_I(\mathcal{P}) < 1 - \epsilon$ . Let  $n$  be such that  $2^{-n+1} > |I| \geq 2^{-n}$ ; thus,  $n \geq n^*$ . We may cover  $I$  with three consecutive basic dyadic intervals  $A, B, C$  of length

$2^{-n}$ . Say  $z \in B$ . Since  $\mathcal{P}$  is relatively thin in  $I$ , but thick in  $B$ , this means that  $\mathcal{P}$  must be thin in  $A$  or  $C$ . This leads to large ‘holes’ arbitrarily close to  $z$  in an appropriate  $\Pi_1^0$  class  $\mathcal{Q}$ , which shows that  $z$  is a porosity point. This class  $\mathcal{Q}$  consists of the basic dyadic intervals where  $\mathcal{P}$  is thick:

$$\mathcal{Q} = [0, 1] - \bigcup \{L : \lambda_L(\mathcal{P}) < 1 - \delta\}$$

where  $\delta = \epsilon/4$  and  $L$  ranges over *open* basic dyadic intervals. We obtain that  $\mathcal{Q}$  is porous at  $z$  via porosity constant  $1/3$ .

*Technical detail:* We have

$$\lambda(\mathcal{P} \cap (A \cup B \cup C)) < 3 \cdot 2^{-n} - \epsilon|I| \leq (3 - \epsilon)2^{-n},$$

while

$$\lambda(\mathcal{P} \cap B) \geq (1 - \delta)2^{-n}.$$

Therefore

$$\lambda(\mathcal{P} \cap (A \cup C)) < (2 - (\epsilon - \delta))2^{-n},$$

and so

$$\lambda(\mathcal{P} \cap A) < (1 - (\epsilon - \delta)/2)2^{-n} \text{ or } \lambda(\mathcal{P} \cap C) < (1 - (\epsilon - \delta)/2)2^{-n}.$$

Thus, since  $\frac{3}{8}\epsilon > \delta$  one of  $A, C$  will be removed from  $\mathcal{Q}$ .

The case that  $z \in A$  or  $z \in C$  is similar.  $\square$

**10.2. Background from analysis, and two lemmas on comparing derivatives.** We need notation and a few definitions, mostly taken from [7] or [5]. For a function  $f: \subseteq \mathbb{R} \rightarrow \mathbb{R}$ , the *slope* at a pair  $a, b$  of distinct reals in its domain is

$$S_f(a, b) = \frac{f(a) - f(b)}{a - b}.$$

For an interval  $A$  with endpoints  $a, b$ , we also write  $S_f(A)$  instead of  $S_f(a, b)$ . For a string  $\sigma$  by  $[\sigma]$  we denote the closed basic dyadic interval  $[0.\sigma, 0.\sigma + 2^{-|\sigma|}]$ . The open basic dyadic interval is denoted  $(\sigma)$ . We write  $S_f([\sigma])$  with the expected meaning.

If  $z$  is in an open neighborhood of the domain of  $f$ , the *upper* and *lower derivatives* of  $f$  at  $z$  are

$$\overline{D}f(z) = \limsup_{h \rightarrow 0} S_f(z, z + h) \quad \text{and} \quad \underline{D}f(z) = \liminf_{h \rightarrow 0} S_f(z, z + h),$$

where as usual,  $h$  ranges over positive and negative values. The derivative  $f'(z)$  exists if and only if these values are equal and finite. We can also consider the upper and lower *pseudo*-derivatives defined by:

$$\begin{aligned} \widetilde{D}f(x) &= \liminf_{h \rightarrow 0^+} \{S_f(a, b) \mid a \leq x \leq b \wedge 0 < b - a \leq h\}, \\ \widetilde{D}f(x) &= \limsup_{h \rightarrow 0^+} \{S_f(a, b) \mid a \leq x \leq b \wedge 0 < b - a \leq h\}. \end{aligned}$$

where  $a, b$  range over rationals in  $[0, 1]$ . We only use them because in our arguments it is often convenient to consider (rational) intervals containing  $x$ , rather than intervals with  $x$  as an endpoint. Also, we want to be able to discuss pseudo-differentiability for partial functions that are defined on all rationals in  $[0, 1]$ , such as in the last section of [7].

Brattka et al. [7, after Fact 2.4] check that  $\underline{D}f(z) \leq \underline{D}f(z) \leq \tilde{D}f(z) \leq \overline{D}f(z)$  for any real  $z \in [0, 1]$ ; in [7, Fact 7.2] they verify that for continuous functions with domain  $[0, 1]$ , the lower and upper pseudo-derivatives of  $f$  coincide with the usual lower and upper derivatives.

They also coincide if  $f$  is nondecreasing: for instance, to show  $\underline{D}f(z) \leq \underline{D}f(z)$ , fix an arbitrarily small  $\epsilon > 0$ . Given  $h > 0$ , choose rationals  $a \leq z$ ,  $z + h \leq b$  such that  $(b - a) \leq (1 + \epsilon)h$ . Then  $S_f(z, z + h) \leq (1 + \epsilon)S_f(a, b)$ .

We will use the subscript 2 to indicate that all the limit operations are restricted to the case of basic dyadic intervals containing  $z$ . For instance,

$$\tilde{D}_2 f(x) = \limsup_{|A| \rightarrow 0} \{S_f(A) \mid x \in A \wedge A \text{ is basic dyadic interval}\}.$$

10.2.1. *A pair of analytical lemmas.* Similar to Theorem 10.5, we show that discrepancy of dyadic and full upper/lower derivatives at  $z$  implies that some closed set is porous at  $z$ .

**Lemma 10.7.** *Suppose  $f: [0, 1] \rightarrow \mathbb{R}$  is a nondecreasing function. Suppose for a real  $z \in [0, 1]$ , with binary representation  $z = 0.Z$ , there is rational  $p$  such that*

$$\tilde{D}_2 f(z) < p < \tilde{D}f(z).$$

*Let  $\sigma^* \prec Z$  be any string such that  $\forall \sigma [Z \succ \sigma \succeq \sigma^* \Rightarrow S_f([\sigma]) \leq p]$ . Then the closed set*

$$(1) \quad \mathcal{C} = [\sigma^*] - \bigcup \{(\sigma) : \sigma \succeq \sigma^* \wedge S_f([\sigma]) > p\},$$

*which contains  $z$ , is porous at  $z$ .*

*Proof.* Suppose  $k \in \mathbb{N}$  is such that  $p(1 + 2^{-k+1}) < \tilde{D}f(z)$ . We show that there exists arbitrarily large  $n$  such that some basic dyadic interval  $[a, \dot{a}]$  of length  $2^{-n-k}$  is disjoint from  $\mathcal{C}$ , and contained in  $[z - 2^{-n+2}, z + 2^{-n+2}]$ . In particular, we can choose  $2^{-k-2}$  as a porosity constant.

By choice of  $k$  there is an interval  $I \ni z$  of arbitrarily short positive length such that  $p(1 + 2^{-k+1}) < S_f(I)$ . Let  $n$  be such that  $2^{-n+1} > |I| \geq 2^{-n}$ . Let  $a_0$  be greatest of the form  $v2^{-n-k}$ ,  $v \in \mathbb{Z}$ , such that  $a_0 < \min I$ . Let  $a_v = a_0 + v2^{-n-k}$ . Let  $r$  be least such that  $a_r \geq \max I$ .

Since  $f$  is nondecreasing and  $a_r - a_0 \leq |I| + 2^{-n-k+1} \leq (1 + 2^{-k+1})|I|$ , we have

$$S_f(I) \leq S_f(a_0, a_r)(1 + 2^{-k+1}),$$

and therefore  $S_f(a_0, a_r) > p$ . Then, by the averaging property of slopes at consecutive intervals of equal length, there is an  $u < r$  such that

$$S_f(a_u, a_{u+1}) > p.$$

Since  $(a_u, a_{u+1}) = (\sigma)$  for some string  $\sigma$ , this gives the required ‘hole’ in  $\mathcal{C}$  which is near  $z \in I$  and large on the scale of  $I$ : in the definition of porosity let  $\beta = 2^{-n+2}$  and note that we have  $[a_u, a_{u+1}] \subseteq [z - 2^{-n+2}, z + 2^{-n+2}]$  because  $z \in I$  and  $|I| < 2^{-n+1}$ .  $\square$

There also is a **dual lemma** for lower derivatives. Note that it can *not* simply be obtained from the first by taking  $-f$  because the function in the dual lemma is still nondecreasing. In fact, now the shortish dyadic intervals

we choose in the proof are all *contained in*  $I$ . (So in fact we can get a porosity constant  $2^{-k-1}$ .)

**Lemma 10.8.** *Suppose  $f: [0, 1] \rightarrow \mathbb{R}$  is a nondecreasing function. Suppose for a real  $z \in [0, 1]$ , with binary representation  $z = 0.Z$ , there a rational  $q$  such that*

$$\underline{D}f(z) < q < \underline{D}_2f(z).$$

*Let  $\sigma^* \prec Z$  be any string such that  $\forall \sigma [Z \succ \sigma \succeq \sigma^* \Rightarrow S_f([\sigma]) \geq q]$ . Then the closed set*

$$\mathcal{C} = [\sigma^*] - \bigcup \{(\sigma): \sigma \succeq \sigma^* \wedge S_f([\sigma]) < q\},$$

*which contains  $z$ , is porous at  $z$ .*

*Proof.* The argument is very similar to the previous one. We will show that we can choose as a porosity constant  $2^{-k-1}$  where  $k \in \mathbb{N}$  is such that  $\underline{D}f(z) < q(1 - 2^{-k+1})$ . There is an interval  $I \ni z$  of arbitrarily short positive length such that  $S_f(I) < q(1 - 2^{-k+1})$ . As before, let  $n$  be such that  $2^{-n+1} > |I| \geq 2^{-n}$ . Let  $a_0$  be least of the form  $v2^{-n-k}$ ,  $v \in \mathbb{Z}$ , such that  $a_0 \geq \min(I)$ . Let  $a_v = a_0 + v2^{-n-k}$ . Let  $r$  be greatest such that  $a_r \leq \max(I)$ .

Since  $f$  is nondecreasing and  $a_r - a_0 \geq |I| - 2^{-n-k+1} \geq (1 - 2^{-k+1})|I|$ , we have

$$S_f(I) \geq S_f(a_0, a_r)(1 - 2^{-k+1}),$$

and therefore  $S_f(a_0, a_r) < q$ . Then there is an  $u < r$  such that

$$S_f(a_u, a_{u+1}) < q.$$

As before, this gives the required hole in  $\mathcal{C}$  which is near  $z \in I$ .  $\square$

10.2.2. *Basic dyadic intervals shifted by 1/3.* For  $m \in \mathbb{N}$  let  $\mathcal{D}_m$  be the collection of intervals of the form

$$[k2^{-m}, (k+1)2^{-m}]$$

where  $k \in \mathbb{Z}$ . Let  $\mathcal{D}'_m$  be the set of intervals  $(1/3) + I$  where  $I \in \mathcal{D}_m$ . We use a ‘geometric’ fact from Morayne and Solecki [29]:

**Fact 10.9.** *Let  $m \geq 1$ . If  $I \in \mathcal{D}_m$  and  $J \in \mathcal{D}'_m$ , then the distance between an endpoint of  $I$  and an endpoint of  $J$  is at least  $1/(3 \cdot 2^m)$ .*

To see this: assume that  $k2^{-m} - (p2^{-m} + 1/3) < 1/(3 \cdot 2^m)$ . This yields  $(3k - 3p - 2^m)/(3 \cdot 2^m) < 1/(3 \cdot 2^m)$ , and hence  $3|2^m$ , a contradiction.

In the following we need values of functions at endpoints of any such intervals. So we think of nondecreasing functions  $f: [0, 1] \rightarrow \mathbb{R}$  extended to all of  $\mathbb{R}$  via  $f(x) = f(0)$  for  $x < 0$  and  $f(y) = f(1)$  for  $y > 1$ . Effectiveness properties, such as computable or interval-c.e. (defined below), are preserved by this because it suffices to compute values of the function in question at rationals.

**10.3. Differentiability of nondecreasing computable functions.** We give a short proof of the following.

**Theorem 10.10** ([7], Thm. 4.1). *Suppose  $f: [0, 1] \rightarrow \mathbb{R}$  is a nondecreasing computable function. Let  $z \in [0, 1]$  be computably random. Then  $f'(z)$  exists.*

*Proof.* We may assume  $z > 1/2$ , else we work with  $f(x + 1/2)$  instead of  $f$ .

Recall that a Cauchy name is a sequence  $(p_i)_{i \in \mathbb{N}}$ ,  $p_i \in \mathbb{Q}$ , such that  $\forall k > i \ |p_i - p_k| \leq 2^{-i}$ . Consider the computable martingale

$$M(\sigma) = S_f(0.\sigma, 0.\sigma + 2^{-|\sigma|}).$$

Computability of  $M$  means that  $M(\sigma)$  is given by a uniformly in  $\sigma$  computable Cauchy name. We denote by  $M(\sigma)_u$  the  $u$ -th term of this Cauchy name, so that  $|M(\sigma) - M(\sigma)_u| \leq 2^{-u}$ .

Note that  $\lim_n M(Z \upharpoonright_n)$  exists and is finite for each computably random real  $Z$ . This is a version of Doob martingale convergence; see, for instance [12]. Returning to the language of slopes, the convergence of  $M$  on  $Z$  means that  $\underline{D}_2 f(z) = \tilde{D}_2 f(z) < \infty$ .

Assume for a contradiction that  $f'(z)$  fails to exist.

First suppose that  $\tilde{D}_2 f(z) < \tilde{D} f(z)$ . Choose rationals  $r, p$  such that  $\tilde{D}_2 f(z) < r < p < \tilde{D} f(z)$ . Choose  $u \in \mathbb{N}$  so large that  $\tilde{D}_2 f(z) < r - 2^{-u}$  and  $r + 2^{-u} < p$ . As usual let  $Z \in 2^\mathbb{N}$  be such that  $z = 0.Z$ . Let  $n^*$  be sufficiently large so that  $[S_f(A) \leq r - 2^{-u}]$  for each basic dyadic interval  $A$  containing  $z$  and of length  $\leq 2^{-n^*}$ . Choose  $k$  with  $p(1 + 2^{-k+1}) < \tilde{D} f(z)$ . Then Lemma 10.7 applies via the string  $\sigma^* = Z \upharpoonright_{n^*}$ .

We define a computable rational-valued martingales  $L, L'$  such that  $L$  succeeds on  $Z$ , or  $L'$  succeeds on  $Y$  where  $0.Y$  is the binary expansion of  $z - 1/3$ .

**Defining  $L$ .** It suffices to consider strings  $\sigma \succeq \sigma^*$ . Let  $L(\sigma^*) = 1$ . Suppose  $\eta \succeq \sigma^*$  and  $L(\eta)$  has been defined. Check if there is a string  $\alpha$  of length  $k + 4$  such that  $M(\eta\alpha)_u > r$ . (Note we have an algorithm for that because  $f$  is computable.)

If so, bet 0 on  $\eta\alpha$  (we know that  $\eta\alpha \not\prec Z$ , so this won't make us lose along  $Z$ ). In return, increase the capital by a factor of  $2^{k+4}/(2^{k+4} - 1)$  along all strings  $\eta\hat{\alpha}$  such that  $|\hat{\alpha}| = k + 4$  and  $\hat{\alpha} \neq \alpha$ . Continue the strategy with all strings  $\eta\hat{\alpha}$ .

If no such  $\alpha$  exists, don't bet, that is, let  $L(\eta 0) = L(\eta 1) = L(\eta)$ . Continue with the strings  $\eta 0$  and  $\eta 1$ .

**Defining  $L'$ .** Let  $\rho^* = Y \upharpoonright_{n^*+1}$ . It suffices to consider strings  $\rho \succeq \rho^*$ .

Let  $L'(\rho^*) = 1$ . Suppose  $\rho \succeq \rho^*$  and  $L(\rho)$  has been defined. Check if there is a string  $\beta$  of length  $k + 5$  such that  $[\rho\beta] + 1/3 \subseteq [\tau]$  for a string  $\tau$  of length  $|\rho\beta| - 1$ , and  $M(\tau)_u > r$ .

If so, bet 0 on  $\rho\beta$  (we know that  $\rho\beta \not\prec Y$ ). In return, increase the capital by a factor of  $2^{k+5}/(2^{k+5} - 1)$  along all strings  $\rho\hat{\beta}$  such that  $|\hat{\beta}| = k + 5$  and  $\hat{\beta} \neq \beta$ . Continue the strategy with all strings  $\rho\hat{\beta}$ .

If no such  $\beta$  exists, don't bet, that is, let  $L(\rho 0) = L(\rho 1) = L(\rho)$ . Continue with the strings  $\rho 0$  and  $\rho 1$ .



We show that  $L$  succeeds on  $Z$ , or  $L'$  succeeds on  $Y$ . Let  $\mathcal{C}$  be the class from (1) in Lemma 10.7. Consider  $n \geq n^* + 4$  and a hole  $[a, \dot{a}] \cap \mathcal{C} = \emptyset$  where  $[a, \dot{a}]$  is a basic dyadic interval of length  $2^{-n-k}$ , and  $[a, \dot{a}] \subseteq [z - 2^{-n+2}, z + 2^{-n+2}]$ .

By Fact 10.9 we have

**Claim 10.11.** *One of the following is true.*

- (i)  $z, a, \dot{a}$  are all contained in a single interval  $A$  taken from  $\mathcal{D}_{n-4}$ .
- (ii)  $z, a, \dot{a}$  are all contained in a single interval  $A'$  taken from  $\mathcal{D}'_{n-4}$ .

In case (i) let  $A = [\eta]$ , so that  $\eta \prec Z$  (recall  $Z \notin \mathbb{Q}$  so there is no problem with the end points). Let  $[a, \dot{a}] = \eta\alpha$  where  $|\alpha| = k + 4$ . We have  $z \notin [a, \dot{a}]$ , and  $L$  increases its capital by a factor of  $2^{k+4}/(2^{k+4} - 1)$  along all strings  $\eta\hat{\alpha}$  as above.

Now suppose case (ii) applies. Let  $\rho$  be the string such that  $A' = [\rho] + 1/3$ . There is  $[b, \dot{b}]$  from  $\mathcal{D}'_{n+k+1}$  with  $[b, \dot{b}] \subseteq [a, \dot{a}]$ . Since (ii) holds we have  $[b, \dot{b}] = [\rho\beta]$  for some string  $\beta$  of length  $k + 5$ . We have  $z \notin [b, \dot{b}]$  and  $L'$  increases its capital by a factor of  $2^{k+5}/(2^{k+5} - 1)$  along all strings  $\rho\hat{\beta}$  as above.

Suppose now that  $L$  fails on  $Z$ . Then for all sufficiently long  $\gamma \prec Y$  we can find  $\rho$  with  $\gamma \preceq \rho \prec Z$  and  $L'$  increases its capital by a fixed factor  $> 1$  on the next  $k + 5$  bits of  $Y$ . Also the capital of  $L'$  along  $Y$  never decreases, because there is no basic dyadic interval  $[\tau] \ni z$  with  $|\tau| \geq n^*$  and  $S_f(\tau)_u \geq r$ . So  $L'$  succeeds on  $Y$ .

The case  $\underline{D}f(z) < \underline{D}_2f(z)$  is analogous, using Lemma 10.8 instead of Lemma 10.7.  $\square$

The method of the proof has an interesting consequence. See e.g. [30, 7.6.2] or [12] for the definition of Church (or computable) stochasticity. By [1], also see [12, 6.4.11],  $X \in 2^{\mathbb{N}}$  is Church stochastic iff no computable martingale that uses only finitely many, positive rational betting factors can win on  $X$ . The martingales  $L, L'$  constructed above are of this kind (in fact we have to modify them slightly in order to avoid betting 0).

**Corollary 10.12.** *Suppose that  $z$  is Church stochastic. Then for each non-decreasing computable function  $f: [0, 1] \rightarrow \mathbb{R}$ , we have  $\tilde{D}_2f(z) = \tilde{D}f(z)$  and  $\underline{D}_2f(z) = \underline{D}f(z)$ .*

This means that on the rather generous class of Church stochastic reals  $z$ , the lower/upper derivative of a nondecreasing computable  $f$  is completely given by the slopes at basic dyadic intervals containing  $z$ . In particular, the derivative at  $z$  equals the dyadic derivative.

**10.4. Polynomial time randomness and differentiability.** Recall that we represent a real  $x$  by a Cauchy name  $(p_i)_{i \in \mathbb{N}}$ . We have  $p_i \in \mathbb{Q}$ , and  $\forall k > i |p_i - p_k| \leq 2^{-i}$ . For feasible analysis, we use a compact set of Cauchy names: the signed digit representation of a real. Such Cauchy names, called *special*, have the form  $p_i = \sum_{k=0}^i b_k 2^{-k}$ , where  $b_k \in \{-1, 0, 1\}$ . (Also,  $b_0 = 0, b_1 = 1$ .) So they are given by paths through  $\{-1, 0, 1\}^\omega$ , something a resource bounded TM can process. We call the  $b_k$  the *symbols* of the special Cauchy name.

**Definition 10.13.** A function  $g: [0, 1] \rightarrow \mathbb{R}$  is polynomial time computable if there is a polynomial time TM turning every special Cauchy name for  $x \in [0, 1]$  into a special Cauchy name for  $g(x)$ .

This means that the first  $n$  symbols of  $g(x)$  can be computed in time  $\text{poly}(n)$ , thereby using polynomially many symbols of the oracle tape holding  $x$ . Functions such as  $e^x, \sin x$  are polynomial time computable, essentially because analysis gives us rapidly converging approximation sequences, such as  $\sum x^n/n!$ .

The argument given above can be adapted to polynomial time randomness. A martingale  $M: 2^{<\omega} \rightarrow \mathbb{R}$  is called polynomial time computable if from string  $\sigma$  and  $i \in \mathbb{N}$  we can in time polynomial in  $|\sigma| + i$  compute the  $i$ -th component of a special Cauchy name for  $M(\sigma)$ . In this case we can compute a polynomial time rational valued martingale dominating  $M$  (Schnorr / Figueira-N). We say  $Z$  is *polynomial time random* if no polynomial time martingale succeeds on  $Z$ . For definitions omitted here see [14].

**Theorem 10.14.** *Let  $z \in [0, 1]$ . Then  $z$  is polynomial time random  $\Leftrightarrow f'(z)$  exists for each nondecreasing polynomial time computable function  $f: [0, 1] \rightarrow \mathbb{R}$ .*

The implication  $\Rightarrow$  and other results were independently proved by A. Kawamura, who directly adapted the proof of [7], Thm. 4.1] to the polynomial time setting.

*Proof.*  $\Leftarrow$ : Suppose  $z$  is not polynomial time random. Then some polynomial time martingale  $L$  succeeds on the binary expansion  $Z$  of  $z$ . By [14, Lemma 3], there is a polynomial time martingale  $M$  with the savings property that succeeds on  $Z$ . Let  $\mu_M$  be the corresponding measure given by  $\mu_M([\sigma]) = 2^{-|\sigma|}M(\sigma)$ . Let  $\text{cdf}_M$  be the cumulative distribution function of  $\mu_M$  given by  $\text{cdf}_M(x) = \mu_M[0, x]$ . By [14, Lemma 3], for each dyadic rational  $p$ ,  $\text{cdf}_M(p)$  is a dyadic rational that can be computed from  $p$  in polynomial time. Since  $M$  has the savings property, by [14, Prop. 5],  $\text{cdf}_M$  satisfies the ‘almost Lipschitz condition’: there is  $\epsilon > 0$  such that for every  $x, y \in [0, 1]$ , if  $y - x \leq \epsilon$  then

$$\text{cdf}_M(y) - \text{cdf}_M(x) = O(-(y - x) \cdot \log(y - x)).$$

This implies that  $f = \text{cdf}_M$  is polynomial time computable: Suppose we are given a special Cauchy name  $(p_i)_{i \in \mathbb{N}}$  for a real  $z$ . We know that  $|z - p_{n+\log n}| = O(2^{-n-\log n})$ . So by the pseudo Lipschitz condition, we have  $|f(z) - f(p_{n+\log n})| = O(2^{-n})$ . So a TM can determine in polynomial time from the first  $n + \log n$  symbols of the special Cauchy name for  $z$  the first  $n$  symbols of a special Cauchy name for  $f(z)$ .

$\Rightarrow$ : Since  $f$  is polynomial time computable, all the martingales involved in the proof of Theorem 10.10 are computable in polynomial time. The usual proof of Doob martingale convergence can be turned into a polynomial time construction, and hence shows that any polynomial time martingale converges on every polynomial random real. Thus we have  $\underline{D}_2 f(z) = \bar{D}_2 f(z) < \infty$ . Furthermore, by the base invariance of polynomial time randomness [14, Thm. 4], if  $z$  is polynomially random then so is  $z - 1/3$ . So  $\tilde{D}_2 f(z) = \tilde{D} f(z)$  and  $\underline{D} f(z) = \underline{D}_2 f(z)$  by the argument given above.  $\square$

### 10.5. Interval c.e. functions.

10.5.1. *Background.* We quote from [2]. Let  $g: [0, 1] \rightarrow \mathbb{R}$ . For  $0 \leq x < y \leq 1$  define the *variation* of  $g$  in  $[x, y]$  by

$$V(g, [x, y]) = \sup \left\{ \sum_{i=1}^{n-1} |g(t_{i+1}) - g(t_i)| : x \leq t_1 \leq t_2 \leq \dots \leq t_n \leq y \right\}.$$

The function  $g$  is of bounded variation if  $V(g, [0, 1])$  is finite. If  $g$  is a continuous function of bounded variation then the function  $f(x) = V(g, [0, x])$  is also continuous. If  $g$  is computable then the function  $f(x) = V(g, [0, x])$  is lower semicomputable (but may fail to be computable). A further property of this “variation function” comes from the observation that  $V(g, [x, y]) + V(g, [y, z]) = V(g, [x, z])$  for  $x < y < z$  (see [6, Prop. 5.2.2]).

Identifying the variations of computable functions, Freer, Kjos-Hanssen, Nies and Stephan [17] studied a class of monotone, continuous, lower semicomputable functions which they called *interval-c.e.*

**Definition 10.15.** A non-decreasing, lower semicontinuous function  $f: [0, 1] \rightarrow \mathbb{R}$  is *interval-c.e.* if  $f(0) = 0$ , and  $f(y) - f(x)$  is a left-c.e. real, uniformly in rationals  $x < y$ .

Thus, the variation function of each computable function of bounded variation is interval-c.e. Freer et al. [17], together with Rute, showed that conversely, every continuous interval-c.e. function is the variation of a computable function. (End quote.)

Note that the better term would be *interval-left-c.e.* There is also a dual concept, being *interval right-c.e.*, where  $f(y) - f(x)$  is a uniformly a right-c.e. real. For instance, the function  $f(x) = \lambda([0, x] \cap \mathcal{P})$  for an effectively closed class  $\mathcal{P}$  is interval right-c.e. There is a curious break of symmetry that the variations of computable functions are the continuous interval *left-c.e.* functions vanishing at 0. This seems to say the left-c.e. version is the cooler one.

(We note that either class is closed under the ‘double mirror’ transformation: if  $f$  is interval left-c.e. [right c.e.] then so is  $\hat{f}(x) = 1 - f(1 - x)$ . The slopes  $S_{\hat{f}}(x, y) = S_f(1 - y, 1 - x)$ .)

10.5.2. *Interval (left)-c.e. functions: upper dyadic equals upper full derivative for non-porosity points.*

**Proposition 10.16.** Let  $f: [0, 1] \rightarrow \mathbb{R}$  be interval-c.e. Then  $\tilde{D}_2 f(z) = \tilde{D} f(z)$  for each non-porosity point  $z$ .

*Proof.* Assume  $\tilde{D}_2 f(z) < \tilde{D} f(z)$ . Since  $f$  is interval c.e., we can view  $S_f(\sigma)$  as a left-c.e. martingale. In particular, the class  $\mathcal{C}$  defined in (1) in Lemma 10.7 is effectively closed. This class is porous at  $z$  for a contradiction.  $\square$

10.5.3. *Dual fact for interval right-c.e. functions.*

**Remark 10.17.** If  $f$  is interval right-c.e. we can apply the dual Lemma 10.8 to conclude that,  $\underline{D} f(z) = \underline{D}_2 f(z)$  for each non-porosity point  $z$ . For instance, let  $f$  be the Lipschitz function given by  $f(x) = \lambda([0, x] \cap \mathcal{P})$  for an

effectively closed class  $\mathcal{P}$ . Then we may conclude that (lower) dyadic density of  $\mathcal{P}$  at a non-porosity point  $x$  coincides with the (lower) full density, thereby obtaining a strengthening of Proposition 10.6.

10.5.4. *Interval c.e. functions: dyadic equals full derivative for reals at which all left-c.e. martingales converge.* Consider a real  $z \in [0, 1] - \mathbb{Q}$ . If a martingale  $M$  converges to a finite value at the binary expansion of  $z$ , we write  $M(z)$  for this finite value. We say that  $z$  is a *convergence point for c.e. martingales* if  $M(z)$  exists for each c.e. martingale  $M$ .

Convergence points for c.e. martingales coincide with the ML-random (dyadic) density one points. This was obtained by 2012 work of a group in Madison consisting of Uri Andrews, Mingzhong Cai, David Diamondstone, Steffen Lempp, and Joseph S. Miller. The implication

$$\text{martingale convergence} \Rightarrow \text{density one}$$

was already pointed out in [2]. The hard implication is

$$\text{dyadic density one} \Rightarrow \text{martingale convergence}.$$

See Theorem 13.1 below.

**Theorem 10.18.** *Let  $f: [0, 1] \rightarrow \mathbb{R}$  be interval-c.e. Let  $z$  be a convergence point for c.e. martingales. Then  $f'(z)$  exists.*

*Proof.* We may assume  $z > 1/2$ , else we work with  $f(x + 1/2)$  instead of  $f$ . The real  $z$  is a dyadic density one point, hence a (full) density one point by the Khan-Miller Theorem 10.5. Then  $z - 1/3$  is also a ML-random density-one point, so using the work of the Madison group discussed in Section 13,  $z - 1/3$  is also a c.e. martingale convergence point. In particular, both  $z$  and  $z - 1/3$  are non-porosity points.

For a nondecreasing function  $g: [0, 1] \rightarrow \mathbb{R}$  recall that  $M_g$  is the (dyadic) martingale associated with the slope  $S_g$  evaluated at intervals of the form  $[i2^{-n}, (i+1)2^{-n}]$ . Thus,

$$M_g(\sigma) = S_g(0.\sigma, 0.\sigma + 2^{-|\sigma|}).$$

Let  $M = M_f$ . Note that  $M$  converges on  $z$  by hypothesis. Thus  $\underline{D}_2 f(z) = \tilde{D}_2 f(z) = M(z)$ .

By Proposition 10.16 again, we have  $\tilde{D}_2 f(z) = \tilde{D} f(z)$ . It remains to show that

$$(2) \quad \underline{D} f(z) = \underline{D}_2 f(z).$$

Since  $f$  is nondecreasing, this will establish that  $f'(z)$  exists.

Let  $\hat{f}(x) = f(x + 1/3)$ , and let  $M' = M_{\hat{f}}$ . We now show that  $M'$  converges on  $z - 1/3$ , and the limits coincide.

**Claim 10.19.**  $M(z) = M'(z - 1/3)$ .

As pointed out above,  $z - 1/3$  is also a convergence point for c.e. martingales. So  $M'$  converges on  $z - 1/3$ . If  $M(z) < M'(z - 1/3)$  then  $\tilde{D}_2 f(z) < \tilde{D} f(z)$ . However  $z$  is a non-porosity point, so this contradicts Proposition 10.16. If  $M'(z - 1/3) < M(z)$  we argue similarly using that  $z - 1/3$  is a non-porosity point. This establishes the claim. Hooray!

To show (2), we extend the method in the proof of Lemma 10.8, taking into account both dyadic intervals, and dyadic intervals shifted by  $1/3$ . Recall that  $\underline{D}_2 f(z) = M(z)$ . Assume for a contradiction that (2) fails. Then we can choose rationals  $p, q$  such that

$$\underline{D}f(z) < p < q < M(z) = M'(z - 1/3).$$

Let  $k \in \mathbb{N}$  be such that  $p < q(1 - 2^{-k+1})$ . Let  $u, v$  be rationals such that

$$q < u < M(z) < v \text{ and } v - u \leq 2^{-k-3}(u - q).$$

Let  $n^* \in \mathbb{N}$  be such that for each  $n \geq n^*$  and any interval  $A \in \mathcal{D}_n \cup \mathcal{D}'_n$ , we have  $S_f(A) \geq u$ .

Let

$$\begin{aligned} \mathcal{E} &= \{X \in 2^{\mathbb{N}} : \forall n \geq n^* M(X \upharpoonright_n) \leq v\} \\ \mathcal{E}' &= \{W \in 2^{\mathbb{N}} : \forall n \geq n^* M'(W \upharpoonright_n) \leq v\} \end{aligned}$$

Since  $f$  is interval c.e., these classes are  $\Pi_1^0$ . In Cantor space we can apply notions of porosity via the usual transfer to  $[0, 1]$  given by the binary expansion.

Let  $0.Z$  be as usual the binary expansion of  $z$ . By the choice of  $n^*$  we have  $Z \in \mathcal{E}$ . Let  $0.Y$  be the binary expansion of  $z - 1/3$ . We have  $Y \in \mathcal{E}'$ .

We will show that  $\mathcal{E}$  is porous at  $Z$ , or  $\mathcal{E}'$  is porous at  $Y$ .

Consider an interval  $I \ni z$  of positive length  $\leq 2^{-n^*-3}$  such that  $S_f(I) \leq p$ . Let  $n$  be such that  $2^{-n+1} > |I| \geq 2^{-n}$ . Let  $a_0 [b_0]$  be least of the form  $w2^{-n-k} [w2^{-n-k} + 1/3]$ , where  $w \in \mathbb{Z}$ , such that  $a_0[b_0] \geq \min(I)$ . Let  $a_i = a_0 + i2^{-n-k}$  and  $b_j = b_0 + j2^{-n-k}$ . Let  $r, s$  be greatest such that  $a_r \leq \max(I)$  and  $b_s \leq \max(I)$ .

As before, since  $f$  is nondecreasing and  $a_r - a_0 \geq |I| - 2^{-n-k+1} \geq (1 - 2^{-k+1})|I|$ , we have  $S_f(I) \geq S_f(a_0, a_r)(1 - 2^{-k+1})$ , and therefore  $S_f(a_0, a_r) < q$ . Then there is an  $i < r$  such that  $S_f(a_i, a_{i+1}) < q$ . Similarly, there is  $j < s$  such that  $S_f(b_j, b_{j+1}) < q$ .

**Claim 10.20.** *One of the following is true.*

- (i)  $z, a_i, a_{i+1}$  are all contained in a single interval taken from  $\mathcal{D}_{n-3}$ .
- (ii)  $z, b_j, b_{j+1}$  are all contained in a single interval taken from  $\mathcal{D}'_{n-3}$ .

For suppose that (i) fails. Then there an endpoint of an  $A \in \mathcal{D}_{n-3}$  (that is, a number of the form  $w2^{-n+3}$  with  $w \in \mathbb{Z}$ ) between  $\min(z, a_i)$  and  $\max(z, a_{i+1})$ . Note that  $\min(z, a_i)$  and  $\max(z, a_{i+1})$  are in  $I$ . By Fact 10.9 and  $|I| < 2^{-n+1}$ , there can be no endpoint of an interval  $A' \in \mathcal{D}'_{n-3}$  in  $I$ . Then, since  $b_j, b_{j+1} \in I$ , (ii) holds. This establishes the claim.

Suppose  $I$  is an interval as above and  $2^{-n+1} > |I| \geq 2^{-n}$ , where  $n \geq n^* + 3$ . Let  $\eta = Z \upharpoonright_{n-3}$  and  $\eta' = Y \upharpoonright_{n-3}$ .

If (i) holds for this  $I$  then there is a string  $\alpha$  of length  $k + 3$  (where  $[\eta\alpha] = [a_i, a_{i+1}]$ ) such that  $M(\eta\alpha) < q$ . So by the choice of  $q < u < v$  and since  $M(\eta) \geq u$  there is  $\beta$  of length  $k + 3$  such that  $M(\eta\beta) > v$ . This yields a hole in  $\mathcal{E}$ , large and near  $Z$  on the scale of  $I$ , which is required for porosity of  $\mathcal{E}$  at  $Z$ .

Similarly, if (ii) holds for this  $I$ , then there is a string  $\alpha$  of length  $k + 3$  (where  $[\eta'\alpha] = [b_j, b_{j+1}]$ ) such that  $M(\eta'\alpha) < q$ . So by the choice of  $q <$

$u < v$  and since  $M'(\eta') \geq u$  there is a string  $\beta$  of length  $k + 3$  such that  $M'(\eta'\beta) > v$ . This yields a hole large and near  $Y$  on the scale of  $I$  required for porosity of  $\mathcal{E}'$  at  $Y$ .

Thus, if case (i) applies for arbitrarily short intervals  $I$ , then  $\mathcal{E}$  is porous at  $Z$ , whence  $z$  is a porosity point. Otherwise (ii) applies for intervals below a certain length. Then  $\mathcal{E}'$  is porous at  $Y$ , whence  $z - 1/3$  is a porosity point.  $\square$

#### 10.5.5. Interval c.e. functions: characterizing ML-randomness.

Nies and Stephan have shown that there is an interval c.e. function  $h$  whose points of differentiability coincides with the ML-randoms. The same is true for the convergence points of the left-c.e. martingale  $S_h(\sigma)$ . All this is obtained from the following stronger statement, which also strengthens [4, Cor.6.6]:

**Theorem 10.21.** *There is a continuous interval c.e. function  $h$  such that  $h'(x)$  exists for each ML-random real  $x$ , and  $\underline{D}h(x) = \infty$  whenever  $x$  is not ML-random.*

*Proof.* Brattka et al. [7, Lemma 6.5] show that there is a computable function  $f$  of bounded variation (in fact, absolutely continuous) such that  $f'(z)$  exists only for Martin-Löf random reals  $z$ . Let

$$h(x) = V(f, [0, x]).$$

To see that  $h$  is as required, we have to look at the construction of  $f$ , which is actually given in [7, proof of Thm. 6.1], a result on weak 2-randomness. The function  $f$  is a superposition of steeper and steeper sawtooth functions based on intervals  $C_{m,i}$  of length rapidly decreasing in  $m$ , which are enumerated into a universal ML test  $\langle \mathcal{G}_m \rangle$ . If  $x$  is ML-random then  $x \notin \mathcal{G}_m$  for almost every  $m$ , and hence for each  $i$  we have  $x \notin C_{m,i}$ . This means that  $h$  is polygonal in a sufficiently small neighbourhood of  $x$ , and  $x$  is not a break point. So  $h'(x)$  exists.

On the other hand, if  $x$  is not ML-random then the change in variation due to the infinite superposition of sawteeth above  $x$  adds up, and so  $\underline{D}h(x) = \infty$ . (Save the amazon.) For detail see the hopefully forthcoming paper [21].  $\square$

## 11. KHAN: A DYADIC DENSITY-ONE POINT THAT IS NOT FULL DENSITY-ONE

(Submitted by Mushfeq Khan, with acknowledgements to Joe Miller for many helpful discussions.)

It seems intuitively likely that being full density-one is a stronger property than being dyadic density-one (see Section 10 for definitions). After all, in the case of the latter, we are severely limiting the types of intervals with which we can witness drops in density. In this section, we construct a dyadic density-one point which is not a full density-one point.

We use the symbol  $\mu$  to refer exclusively to the standard Lebesgue measure on Cantor space. If  $\sigma$  is a string, and  $C$  a measurable set, the shorthand  $\mu_\sigma(C)$  denotes the relative measure of  $C$  in the cone above  $\sigma$ . The following lemma, which is a critical part of the argument, is a special case of the

Kolmogorov inequality for martingales (see for example, [30, 7.1.9], and consider the martingale  $S(\sigma) = \mu_\sigma(W)$ ).

**Lemma 11.1.** *Suppose  $W \subseteq 2^\omega$  is open. Then for any  $\varepsilon$  such that  $\mu(W) \leq \varepsilon \leq 1$ , let  $U_\varepsilon$  denote the set  $\{X \in 2^\omega : \mu_\rho(W) \geq \varepsilon \text{ for some } \rho \prec X\}$ . We call  $U_\varepsilon$  the  $\varepsilon$ -vicinity of  $W$ . Then  $\mu(U_\varepsilon) \leq \mu(W)/\varepsilon$ .*

**Theorem 11.2.** *There is a dyadic density-one point that is not a density-one point.*

*Proof.* We build the desired real  $Y$  by computable approximation. At each stage  $s$  of the construction, we have a sequence of finite strings  $\sigma_{0,s} \prec \sigma_{1,s} \prec \dots$  approximating  $Y$ . At the same time, we build a  $\Sigma_1^0$  class  $B$  whose complement witnesses the fact that  $Y$  is not a density-one point. Let  $W_e$  denote the upward closure of the  $e$ -th c.e. set. Each c.e. set represents a requirement that needs to be met by  $Y$ . In other words, for each  $e$ , if  $Y$  is not in  $[W_e]$ , we require that  $\lim_{\rho \prec Y} \mu_\rho([W_e]) = 0$ . Priorities are assigned to c.e. sets in the usual manner, with  $W_j$  having higher priority than  $W_i$  for any  $i > j$ . We make use of the following shorthand: Let  $C$  be a measurable set and  $\tau$  and  $\tau'$  two strings such that  $\tau \prec \tau'$ . If for every  $\rho$  such that  $\tau \preceq \rho \prec \tau'$ ,  $\mu_\rho(C) < \alpha$ , then we say that *between  $\tau$  and  $\tau'$ ,  $\mu(C) < \alpha$* .

At any stage  $s$ , for each  $k$ , we will be working above  $\sigma_{k,s}$  to define  $\sigma_{k+1,s}$ . We have two goals in mind: Firstly, for any  $e < k$  such that  $\sigma_{k,s}$  is not already a member of  $W_e$ , we must keep the measure of  $W_e$  between  $\sigma_{k,s}$  and  $\sigma_{k+1,s}$  below a certain threshold. If the threshold is exceeded, say at a string  $\rho$  between  $\sigma_{k,s}$  and  $\sigma_{k+1,s}$ , we shall reroute  $\sigma_{k+1}$  above  $\rho$  to enter  $W_e$ . Secondly, we must ensure that there is an interval  $I \subseteq [\sigma_{k,s}]$  such that  $[\sigma_{k+1,s}] \subseteq I$  and  $\mu_I(B) > 1/4$ . Both goals must be satisfied while keeping  $Y$  from entering  $[B]$ . Globally, we must maintain the fact that between  $\sigma_{k,s}$  and  $\sigma_{k+1,s}$ , the measure of  $B$  remains *strictly below* a threshold  $\beta(k, s)$ , which is updated each time we act above  $\sigma_{k,s}$  by rerouting  $\sigma_{k+1}$ . The construction begins by setting  $\sigma_{0,0}$  equal to the empty string.

Process above  $\sigma_{k,s}$ . When we first start working above  $\sigma_{k,s}$ , say at stage  $s_0$ , we set  $\beta(k, s_0) = \beta^*(k)$  (see below for how  $\beta^*(k)$  is defined). If  $k > 0$ , then we start by choosing a  $\nu \succ \sigma_{k,s_0}$  long enough so that between  $\sigma_{k-1,s_0}$  and  $\sigma_{k,s_0}$ ,  $\mu(B \cup [\nu]) < \beta_{k-1,s_0}$ . We let  $\sigma_{k+1,s_0} = \nu 10^j$  and enumerate the string  $\nu 01^j$  into  $B$ , where  $j$  is chosen large enough so that the measure of  $[B]$  between  $\sigma_{k,s_0}$  and  $\sigma_{k+1,s_0}$  remains below  $\beta^*(k)$ . If  $k = 0$ ,  $\nu$  can be chosen to be the empty string.

In a subsequent stage  $s$ , suppose that  $C_0, \dots, C_l$  are those among the first  $k$  c.e. sets that  $\sigma_{k,s}$  is not already a member of, in order of descending priority. Now if for some  $\rho$  between  $\sigma_{k,s}$  and  $\sigma_{k+1,s}$  and some  $j \leq l$ ,  $\mu_\rho([C_j])$  exceeds  $\sqrt{\beta(k, s)}$  and no action has yet been taken for a higher priority  $C_{j'}$ , then we act by rerouting  $\sigma_{k+1,s}$  above  $\rho$ . Let  $\nu \succeq \rho$  be a string in  $C_j$  long enough so that:

- (1) Between  $\rho$  and  $\nu$ ,  $\mu([B]) < \sqrt{\beta(k, s)}$ .
- (2)  $[B] \cap [\nu] = \emptyset$ .
- (3) If  $k > 0$ , then between  $\sigma_{k-1,s}$  and  $\sigma_{k,s}$ ,  $\mu(B \cup [\nu])$  must be strictly less than  $\beta(k-1, s)$ .

Let  $j$  be large enough so that between  $\sigma_{k,s}$  and  $\nu$ ,  $\mu(B \cup [\nu 01^j])$  remains strictly below  $\sqrt{\beta(k, s)}$ . We set  $\sigma_{k+1, s+1} = \nu 10^j$  and enumerate  $\nu 01^j$  into  $B$ . Finally, we set  $\beta(k, s+1) = \sqrt{\beta(k, s)}$ .

**Choosing  $\beta^*(k)$ .** We move  $\sigma_{k+1, s+1}$  into  $C_j$  when the following is seen to occur at some stage  $s$ : For some  $\rho$  between  $\sigma_{k,s}$  and  $\sigma_{k+1,s}$ ,  $\mu_\rho([C_j])$  exceeds the measure of the  $\sqrt{\beta(k, s)}$ -vicinity of  $[B]$  above  $\rho$ , i.e., if  $\mu_\rho([C_j]) > \beta(k, s)/\sqrt{\beta(k, s)} > \mu_\rho(B)/\sqrt{\beta(k, s)}$ . If this does not occur, we wish to limit the measure of  $C_j$  to  $2^{-k}$  between  $\sigma_{k,s}$  and  $\sigma_{k+1,s}$ . Each time we act above  $\sigma_{k,s}$ , the value of  $\beta(k, s+1)$  is magnified by a power of  $1/2$ , so we require that  $\beta^*(k)$  satisfy

$$(\beta^*(k))^{1/2^{k+1}} \leq 2^{-k}.$$

### Verification.

**Claim 11.3.** *Unless we act immediately above  $\sigma_{k,s}$ , the measure of  $B$  between  $\sigma_{k,s}$  and  $\sigma_{k+1,s}$  remains strictly below  $\beta(k, s)$ .*

*Proof.* Condition (2) above ensures that if  $\sigma_{k,s}$  is redefined at stage  $s$  due to an action above  $\sigma_{l,s}$  for some  $l < k$ , then  $\mu(B \cap [\sigma_{k,s}]) = 0$ . If we act above  $\sigma_{k+1,s}$ , then condition (3) ensures that  $\mu(B)$  remains below  $\beta(k, s)$  between  $\sigma_{k,s}$  and  $\sigma_{k+1,s}$ . Note that there is a string  $\nu$  such that  $\sigma_{k+1,s} \prec \nu \prec \sigma_{k+2,s}$  and  $\mu(B \cup [\nu]) < \beta(k, s)$  between  $\sigma_{k,s}$  and  $\sigma_{k+1,s}$ . So if we act above  $\sigma_{l,s}$  for some  $l > k+1$ , then we add some measure to  $B$ , but this measure is contained entirely in  $[\nu]$ .  $\square$

**Claim 11.4.** *We can act above  $\sigma_{k,s}$  while satisfying requirements (1) through (3) above.*

*Proof.* By Claim 11.3,  $\mu(B) < \beta(k, s)$  between  $\sigma_{k,s}$  and  $\sigma_{k+1,s}$ . So if at stage  $s$ , for some  $\rho$  between  $\sigma_{k,s}$  and  $\sigma_{k+1,s}$ ,  $\mu(C_j)$  exceeds  $\sqrt{\beta(k, s)}$  then by Lemma 11.1 there is an  $X \in C_j$  extending  $\rho$  such that for every  $\alpha$  such that  $\rho \preceq \alpha \prec X$ ,  $\mu_\alpha(B) < \sqrt{\beta(k, s)}$ . Thus there are arbitrarily long strings extending  $\rho$  satisfying condition (1). Conditions (2) and (3) are met by simply choosing a long enough such string.  $\square$

**Claim 11.5.** *For each  $k \in \omega$ ,  $\sigma_k = \lim_s \sigma_{k,s}$  exists, and  $Y = \bigcup_k \sigma_k$  is total.*

*Proof.* Assume that  $\sigma_{k,s}$  has stabilized by stage  $s$ . Then  $\sigma_{k+1}$  is redefined above  $\sigma_{k,s}$  at most  $k$  times.  $\square$

**Claim 11.6.**  *$Y$  is a dyadic density-one point.*

*Proof.* Suppose that  $Y \notin [W_e]$ . Let  $k$  be large enough so that  $k > e$  and for all  $e' < e$ , if  $Y \in W_{e'}$ , then  $\sigma_k \in W_{e'}$ . For any  $k' > k$ , let  $s$  be large enough so that  $\sigma_{k',s}$  has stabilized. By our choice of  $k$ , we never act above  $\sigma_{k',s}$  for the sake of  $W_{e'}$  for any  $e' < e$ , and by the assumption that  $Y \notin [W_e]$ , we never act for the sake of  $W_e$ . Let  $t > s$  be such that  $\sigma_{k'+1,t}$  has stabilized. For all  $t' > t$ , between  $\sigma_{k',t'}$  and  $\sigma_{k'+1,t'}$ ,  $\mu(W_e)$  does not exceed  $\sqrt{\beta(k', t')}$ , which is always bounded by  $2^{-k}$ .  $\square$

**Claim 11.7.**  *$Y$  is not a density-one point.*



*Proof.* Let  $\sigma_k$  and  $\sigma_{k+1}$  be the final values of  $\sigma_{k,s}$  and  $\sigma_{k+1,s}$  respectively. Then by construction there is a string  $\nu$  such that  $\sigma_k \prec \nu \prec \sigma_{k+1} \prec Y$ , and  $\sigma_{k+1} = \nu 10^j$  for some  $j$  and  $\nu 01^j \in B$ . Let  $l = |\nu| + j + 1$  and let  $I$  be the interval  $(0.\nu 1 - 2^{-l}, 0.\nu 1 + 2^{-l})$ . Since  $Y$  is a dyadic density-one point,  $Y$  is not a rational and so  $Y \in [0.\nu 1, 0.\nu 1 + 2^{-l}) \subset I$ , while the left half of  $I$  belongs entirely to  $B$ .  $\square$

This completes the proof of Theorem 11.2. We note that the construction actually ensures that  $2^{\mathbb{N}} - B$  is porous at  $Y$ .  $\square$

## 12. NIES: UPPER DENSITY AND PARTIAL COMPUTABLE RANDOMNESS

The *upper* (Cantor-space) density of a set  $\mathcal{C} \subseteq 2^{\mathbb{N}}$  at a point  $Z$  is:

$$\bar{\rho}_2(\mathcal{C}|Z) := \limsup_{\sigma \prec Z \wedge |\sigma| \rightarrow \infty} \frac{\lambda(I \cap \mathcal{C})}{|I|},$$

where  $I$  ranges over basic dyadic intervals containing  $z$ . Bienvenu et al. [3, Prop. 5.4] showed that for any effectively closed set  $\mathcal{P}$  and ML-random  $Z \in \mathcal{P}$ , we have  $\bar{\rho}_2(\mathcal{P} | Z) = 1$ ; this implies of course that the upper density in  $\mathbb{R}$  also equals 1.

The following shows that ML-randomness was actually too strong an assumption. The right level seems to be given by the “ugly duckling” notion of partial computable randomness. See [30, Ch. 7] for background. If the measure  $\lambda \mathcal{P}$  is a computable real, then in fact computable randomness of  $Z$  suffices. In that case the full dyadic density is 1.

**Proposition 12.1.** *Let  $\mathcal{P} \subseteq 2^{\mathbb{N}}$  be effectively closed. Let  $Z \in \mathcal{P}$  be partial computably random. Then  $\bar{\rho}_2(\mathcal{P} | Z) = 1$ .*

*Proof.* Suppose there is  $q < 1$  and  $n^*$  such that  $\lambda_{\sigma}(\mathcal{P}) < q$  for each  $\eta \prec Z$  with  $|\eta| \geq n^*$ . We will define a partial computable martingale  $M$  that succeeds on  $Z$ . Let  $M(\eta) = 1$  for all strings  $\eta$  with  $|\eta| \leq n^*$ . Now suppose that  $M(\eta)$  has been defined for a string  $\eta$  of length at least  $n^*$ , but  $M$  is as yet undefined on extensions of  $\eta$ . Search for  $t > |\eta|$  such that

$$2^{-(t-|\eta|)} \# \{ \tau \mid |\tau| = t \wedge [\tau] \cap \mathcal{P}_t = \emptyset \} > 1 - q.$$

If  $t$  is found, bet all the capital existing at  $\eta$  on the strings  $\sigma \succ \eta$  with  $|\sigma| = t$  that are not  $\tau$ 's as above, thereby multiplying the capital by  $1/q$ . Now repeat with all such strings  $\sigma \succ \eta$  of length  $t$ .

The formal definition of  $M$  is as follows (supplied by Jing Zhang). For all  $|\tau| \leq n^*$ ,  $M(\tau) = 1$ . Next we define  $M$  inductively on  $2^{<\omega}$ . Suppose  $M$  has been defined on  $\alpha$  and  $M(\alpha) = \beta$ , let  $t \in \omega$  such that  $t > |\alpha|$  and let  $S = \{ \tau \in 2^t : [\tau] \cap \mathcal{P}_t \}$  and  $r = |S| > 2^{t-|\tau|}(1-q)$ . For each  $\sigma \in 2^t \setminus S$ , define  $M(\sigma) = \frac{1}{q}\alpha$ , and let  $\tau^* \in S$  be the leftmost element and define

$$M(\tau^*) = 2^{t-|\tau|}\alpha - \frac{1}{q}\alpha(2^{t-|\tau|} - r)$$

For any  $\sigma \succ \alpha$  and  $|\sigma| < t$ , define  $M$  accordingly to make  $M$  a martingale.

Next is the verification. First we check that  $\forall \tau \prec Z$ ,  $M(\tau)$  is defined. We verify this inductively. Suppose  $\eta \prec Z$  is already defined. Then by assumption,  $\lambda_{\eta}(\bar{\mathcal{P}}) > (1-q)$ . Therefore, there exists a stage  $t \in \omega$  such

that  $\lambda_\eta(\bar{P}_t) > (1 - q)$ . Thus we have  $\Sigma_{\tau \in 2^t, \eta \preceq \tau} \lambda_\tau(\bar{P}_t) = 2^{t-|\eta|} \lambda_\eta(\bar{P}_t) > 2^{t-|\eta|}(1 - q)$ ; here we use the fact that for any measurable class  $Q \subset 2^\omega$ , the function  $\sigma \mapsto \lambda_\sigma(Q)$  is a martingale. Therefore, we have found such a  $t$  to define a proper extension of  $\eta$ . By induction,  $M$  is defined on  $Z$ . It is easy to see  $Z$  succeeds on  $M$  since every time a new string is defined, the capital becomes  $\frac{1}{q} > 1$  times of the original capital.

Note that  $M$  succeeds on  $Z$  because *all* strings  $\sigma \prec Z$  of length  $\geq n^*$  qualify as possible  $\eta$ 's where  $t$  exists. On the other hand, if  $\eta$  is off  $Z$  then there may be no  $t$ , so  $M$  can be partial.  $\square$

**Question 12.2.** *Is there a computably random  $Z$  in some  $\Pi_1^0$  class  $\mathcal{P}$  so that  $\bar{\rho}_2(\mathcal{P} \mid Z) < 1$  ?*

**Proposition 12.3.** *Let  $\mathcal{P} \subseteq 2^\mathbb{N}$  be effectively closed with  $\lambda\mathcal{P}$  computable. Let  $Z \in \mathcal{P}$  be computably random. Then  $\rho_2(\mathcal{P} \mid Z) = 1$ .*

*Proof.* First we show  $\bar{\rho}_2(\mathcal{P} \mid Z) = 1$ . The easy, but not quite accurate, argument would be that in the construction above, before searching for  $t$ , we ask whether  $\lambda_\eta(\mathcal{P}) < q$ ; only then do we attempt to find  $t$ .

This isn't quite right because " $\lambda_\eta(\mathcal{P}) < q$ " is merely  $\Sigma_1^0$ , even though  $\lambda_\eta(\mathcal{P})$  is a uniformly in  $\eta$  computable real. To amend this, fix  $q' < q$  such that in fact  $\lambda_\sigma(\mathcal{P}) < q'$  for each  $\eta \prec Z$  with  $|\eta| \geq n^*$ . We ask simultaneously

- (1) whether  $\lambda_\eta(\mathcal{P}) > q'$ ; if the positive answer to this  $\Sigma_1^0$  question turns up first we don't bet on extensions of  $\eta$
- (2)  $\lambda_\eta(\mathcal{P}) < q$ ; in this case we bet.

One of the queries must yield an answer.

The computable martingale  $\eta \rightarrow \lambda_\eta(\mathcal{P})$  cannot oscillate along the computably random  $Z$ . Thus, the dyadic density  $\rho_2(\mathcal{P} \mid Z)$  is 1.  $\square$

In fact, Schnorr randomness of  $Z$  is sufficient as a hypothesis in the preceding proposition by deeper work of [34] and [17]. The characteristic function  $1_P$  is  $L_1$ -computable because there is a sequence  $\langle 1_{P_{g(n)}} \rangle_{n \in \mathbb{N}}$ , where  $g$  is a computable function such that  $\lambda(P_{g(n)} - P) \leq 2^{-n}$ . Now use e.g. [34, Theorem 3.15].

### 13. DENSITY-ONE POINTS AND MADISON TESTS (WRITTEN BY NIES)

The following is 2012 work of a group at Madison, consisting of U. Andrews, M. Cai, D. Diamondstone, S. Lempp, and l.n.l. J. S. Miller. The writeup below, due to Nies, is based on discussions with Miller, and Miller's slides for his talks at the Buenos Aires Semester 2013. Technical details in the verifications have been added.

A martingale  $L: 2^{<\omega} \rightarrow \mathbb{R}_0^+$  is called *left-c.e.* if  $L(\sigma)$  is a left-c.e. real uniformly in  $\sigma$ . We focus on convergence of such a martingale along a real  $Z$ , which means that  $\lim_n L(Z \upharpoonright_n)$  exists in  $\mathbb{R}$ . Unlike the case of computable martingales, convergence requires more randomness than boundedness. For instance, let  $\mathcal{U} = [0, \Omega)$ , and let  $L(\sigma) = \lambda(\mathcal{U} \mid [\sigma])$  (as a shorthand we use  $\lambda_\sigma(\mathcal{U})$  for this conditional measure); then the left-c.e. martingale  $L$  is bounded by 1 but diverges on  $\Omega$  because  $\Omega$  is Borel normal.

**Theorem 13.1** (Andrews, Cai, Diamondstone, Lempp and Miller, 2012). *The following are equivalent for a ML-random real  $z \in [0, 1]$ .*

- (i)  $z$  is a dyadic density-one point.
- (ii) Every left-c.e. martingale converges along  $Z$ , where  $0.Z$  is the binary expansion of  $z$ .

Note that by Theorem 10.5,  $z$  is a full density-one point iff  $z$  is a dyadic density-one point. A ML-random satisfying any of these equivalent conditions will be called *density random*.

*Proof.* (ii)  $\rightarrow$  (i) is [3, Cor 5.5].

(i)  $\rightarrow$  (ii). We can work within Cantor space because dyadic density is the same in Cantor space as in  $[0, 1]$ . For  $X \subseteq 2^{\mathbb{N}}$  we define the weight  $\text{wt}(X) = \sum_{\sigma \in X} 2^{-|\sigma|}$ . Let  $\sigma \prec \tau = \{\tau \in 2^{<\omega} : \sigma \prec \tau\}$ . We use a technical test concept that reveals its beauty only after several days of study.

**Definition 13.2.** A *Madison test* is a computable sequence  $\langle U_s \rangle_{s \in \mathbb{N}}$  of strong indices for finite subsets of  $2^{<\omega}$  such that  $U_0 = \emptyset$ , for each stage  $s$  we have  $\text{wt}(U_s) \leq c$  for some constant  $c$ , and for all strings  $\sigma, \tau$ ,

- (a)  $\tau \in U_s - U_{s+1} \rightarrow \exists \sigma \prec \tau [\sigma \in U_{s+1} - U_s]$
- (b)  $\text{wt}(\sigma \prec \cap U_s) > 2^{-|\sigma|} \rightarrow \sigma \in U_s$ .

Note that by (a),  $U(\sigma) := \lim_s U_s(\sigma)$  exists for each  $\sigma$ ; in fact,  $U_s(\sigma)$  changes at most  $2^{|\sigma|}$  times. We say that  $Z$  *fails*  $\langle U_s \rangle_{s \in \mathbb{N}}$  if  $Z \upharpoonright_n \in U$  for infinitely many  $n$ ; otherwise  $Z$  *passes*  $\langle U_s \rangle_{s \in \mathbb{N}}$ .

Note that  $\text{wt}(U_s) \leq \text{wt}(U_{s+1}) \leq 1$ , and  $\text{wt}(U) = \sup_s \text{wt}(U_s)$ . Thus,  $\text{wt}(U)$  is a left-c.e. real.

**Lemma 13.3.** *Let  $Z$  be a ML-random dyadic density-one point. Then  $Z$  passes each Madison test.*

To see this, suppose that  $Z$  fails a Madison test  $\langle U_s \rangle_{s \in \mathbb{N}}$ . We build a ML-test  $\langle \mathcal{S}^k \rangle_{k \in \mathbb{N}}$  such that  $\forall \sigma \in U [\lambda_\sigma(\mathcal{S}^k) \geq 2^{-k}]$ , and therefore  $\bar{\rho}(2^{\mathbb{N}} - \mathcal{S}^k \mid Z) \leq 1 - 2^{-k}$ . Since  $Z$  is ML-random we have  $Z \notin \mathcal{S}^k$  for some  $k$ . So  $Z$  is not a density-one point.

To define the  $\mathcal{S}^k$  we introduce for each  $k, s \in \omega$  and each string  $\sigma \in U_s$  clopen sets  $\mathcal{A}_{\sigma,s}^k \subseteq [\sigma]$  given by uniformly computable strong indices, such that  $\lambda(\mathcal{A}_{\sigma,s}^k) = 2^{-|\sigma|-k}$  for each  $\sigma \in U_s$ . We update these clopen sets at stages  $s$  when  $\sigma \in U_{s+1} - U_s$ . For each  $\tau \succ \sigma$  with  $\tau \in U_s - U_{s+1}$ , put  $\mathcal{A}_{\tau,s}^k$  into an auxiliary clopen set  $\tilde{\mathcal{A}}_{\sigma,s+1}^k$ . Since  $\sigma \notin U_s$ , by (b) we have  $\text{wt}(\sigma \prec \cap U_s) \leq 2^{-|\sigma|}$ , and so inductively  $\lambda(\tilde{\mathcal{A}}_{\sigma,s+1}^k) \leq 2^{-|\sigma|-k}$ . Now to obtain  $\mathcal{A}_{\sigma,s+1}^k$  simply add mass from  $[\sigma]$  to  $\tilde{\mathcal{A}}_{\sigma,s+1}^k$  in order to ensure equality as required.

Let  $\mathcal{S}_t^k = \bigcup_{\sigma \in U_t} \mathcal{A}_{\sigma,t}^k$ . Then  $\mathcal{S}_t^k \subseteq \mathcal{S}_{t+1}^k$  by property (a) of Madison tests. Clearly  $\lambda \mathcal{S}_t^k \leq 2^{-k} \text{wt}(U_t) \leq 2^{-k}$ . So  $\mathcal{S}^k = \bigcup_t \mathcal{S}_t^k$  determines a ML-test. So  $Z \notin \mathcal{S}^k$  for some  $k$ . If  $\sigma \in U$  then by construction  $\mathcal{A}_{\sigma,s}^k$  has measure  $2^{-|\sigma|-k}$  for almost all  $s$ . Thus  $\lambda_\sigma(\mathcal{S}^k) \geq 2^{-k}$  as required. This shows the lemma.

**Lemma 13.4.** *Suppose that  $Z$  passes each Madison test. Then every left-c.e. martingale  $L$  converges along  $Z$ .*

To see this, first we show that  $Z$  is ML-random. Assume otherwise, so  $Z \in \bigcap_m \mathcal{G}_m$  for a ML-test  $\langle \mathcal{G}_m \rangle$ . Let  $\langle G_m \rangle$  be a uniformly c.e. sequence of antichains in  $2^{<\omega}$  with  $[G_m]^\prec = \mathcal{G}_m$ . We define a Madison test  $\langle U_s \rangle$  where strings never leave. Put the empty string  $\langle \rangle$  into  $U_1$ . If  $\sigma \in U_s$ , put all  $\rho \succ \sigma$  with  $\rho \in G_{|\sigma|+1}$  into  $U_{s+1}$ . Clearly for  $s > 0$  we have  $\text{wt}(U_{s+1} - U_s) \leq \text{wt}(U_s - U_{s-1})/2$ . Thus  $\text{wt}(U_s) \leq 1$  for each  $s$ . Also  $Z$  fails  $\langle U_s \rangle$ .

Now let  $L(\sigma) = \sup_s L_s(\sigma)$  where  $\langle L_s \rangle$  is a uniformly computable sequence of martingales and  $L_0 = 0$ .

If  $L$  diverges along  $Z$ , there is  $\varepsilon < L(\langle \rangle)$  with

$$\limsup_n L(Z \upharpoonright_n) - \liminf_n L(Z \upharpoonright_n) > \varepsilon.$$

Since  $Z$  is computably random, for each  $s$  the limit  $\lim_n L_s(Z \upharpoonright_n)$  exists, and is at most  $\liminf_n L(Z \upharpoonright_n)$ . Thus for each  $s$  there are infinitely many  $n$  with  $L(Z \upharpoonright_n) - L_s(Z \upharpoonright_n) > \varepsilon$ . Based on this insight we define a Madison test which  $Z$  fails. Along with the  $U_s$  we define a uniformly computable labelling function  $\gamma_s: U_s \rightarrow \{0, \dots, s\}$ .

*Let  $U_0 = \emptyset$ . For  $s > 0$  we put the empty string  $\langle \rangle$  into  $U_s$  and let  $\gamma_s(\langle \rangle) = 0$ . If already  $\sigma \in U_s$  with  $\gamma_s(\sigma) = t$ , then we also put into  $U_s$  all strings  $\tau \succ \sigma$  that are minimal under the prefix ordering  $\prec$  with  $L_s(\tau) - L_t(\tau) > \varepsilon$ . Let  $\gamma_s(\tau)$  be the least  $r$  with  $L_r(\tau) - L_t(\tau) > \varepsilon$ .*

Note that  $\gamma_s(\tau)$  simply records the greatest stage  $r \leq s$  at which  $\tau$  entered  $U_r$ . We verify that  $\langle U_s \rangle$  is a Madison test. For (a), suppose that  $\tau \in U_s - U_{s+1}$ . Let  $\sigma_0 \prec \sigma_1 \prec \dots \prec \sigma_n = \tau$  be the prefixes of  $\tau$  in  $U_s$ . We can choose a least  $i < n$  such that  $\sigma_{i+1}$  is no longer the minimal extension of  $\sigma_i$  at stage  $s+1$ . Thus there is  $\eta$  with  $\sigma_i \prec \eta \prec \sigma_{i+1}$  and  $L_{s+1}(\eta) - L_{\gamma_s(\sigma_i)}(\eta) > \varepsilon$ . Then  $\eta \in U_{s+1}$  and  $\eta \prec \tau$ , as required.

To verify (b) requires more work.

We fix  $s$  and write  $M_t(\eta)$  for  $L_s(\eta) - L_t(\eta)$ .

**Claim 13.5.** *For each  $\rho \in U_s$ , where  $\gamma_s(\rho) = r$ , we have*

$$2^{-|\rho|} M_r(\rho) \geq \varepsilon \cdot \text{wt}(U_s \cap \rho^\prec).$$

In particular, for  $\rho = \langle \rangle$ , we obtain that  $\text{wt}(U_s)$  is bounded by a constant  $c = L_s(\langle \rangle)/\varepsilon$  as required.

For  $\sigma \in U_s$  and  $n \in \mathbb{N}$  let  $U_s^{\sigma, n}$  be the strings strictly above  $\sigma$  and at a distance to  $\sigma$  of at most  $n$ , that is, the set of strings  $\tau$  such that there is  $\sigma = \sigma_0 \prec \dots \prec \sigma_m = \tau$  on  $U_s$  with  $m \leq n$  and  $\sigma_{i+1}$  a child of  $\sigma_i$  for each  $i < m$ . To establish the claim, we show by induction on  $n$  that

$$2^{-|\rho|} M_r(\rho) \geq \varepsilon \cdot \text{wt}(U_s^{\rho, n}).$$

If  $n = 0$  then  $U_s^{\rho, n}$  is empty so the right hand side is 0. Now suppose that  $n > 0$ . Let  $F$  be the set of immediate successors of  $\rho$  on  $U_s$ . Let  $r_\tau = \gamma_s(\tau)$ . By the inductive hypothesis, we have for each  $\tau \in F$

$$\begin{aligned} (3) \quad 2^{-|\tau|} M_{r_\tau}(\tau) &= 2^{-|\tau|} [(L_{r_\tau}(\tau) - L_r(\tau)) + M_{r_\tau}(\tau)] \\ &\geq 2^{-|\tau|} \cdot \varepsilon + \varepsilon \cdot \text{wt}(U_s^{\tau, n-1}). \end{aligned}$$

Then, taking the sum over all  $\tau \in F$ ,

$$2^{-|\rho|} M_r(\rho) \geq \sum_{\tau \in F} 2^{-|\tau|} M_r(\tau) \geq \varepsilon \cdot \text{wt}(U_s^{\rho,n}).$$

The first inequality is Kolmogorov's inequality for martingales, using that the  $\tau$  form an antichain. For the second inequality we have used (3) and that  $U_s^{\rho,n} = F \cup \bigcup_{\tau \in F} U_s^{\tau,n-1}$ . This completes the induction and shows the claim.

Now, to obtain (b), suppose that  $\text{wt}(U_s \cap \sigma^{\prec}) > 2^{-|\sigma|}$ . We use Claim 13.5 to show that  $\sigma \in U_s$ . Assume otherwise. Let  $\rho \prec \sigma$  be in  $U_s$  with  $|\rho|$  maximal, and let  $r = \gamma_s(\rho)$ . As before, let  $F$  be the prefix minimal extensions of  $\sigma$  in  $U_s$ , and  $r_\tau = \gamma_s(\tau)$ . Then  $L_{r_\tau}(\tau) - L_r(\tau) > \varepsilon$  for  $\tau \in F$ . Since  $\tau \in U_s$ , we can apply the claim to  $\tau$ , so (3) is valid.

Arguing as before, but with  $\sigma$  instead of  $\rho$ , we have

$$2^{-|\sigma|} M_r(\sigma) \geq \sum_{\tau \in F} 2^{-|\tau|} M_r(\tau) \geq \varepsilon \cdot \text{wt}(U_s \cap \sigma^{\prec})$$

(that part of the argument did not use that  $\rho \in U_s$ ). Since  $\text{wt}(U_s \cap \sigma^{\prec}) > 2^{-|\sigma|}$ , this implies that  $M_r(\sigma) > \varepsilon$ . Hence some  $\sigma'$  with  $\rho \prec \sigma' \preceq \sigma$  is in  $U_s$ , contrary to the maximality of  $\rho$ .

This concludes the verification that  $\langle U_s \rangle$  is a Madison test. As mentioned already, for each  $r$  there are infinitely many  $n$  with  $L(Z \upharpoonright_n) - L_r(Z \upharpoonright_n) > \varepsilon$ . This shows that  $Z$  fails this test: suppose inductively that we have  $\sigma \prec Z$  and  $r$  is least such that  $\sigma \in U_t$  for all  $t \geq r$  (so that  $\gamma_t(\sigma) = r$  for all such  $t$ ). Choose  $n > |\sigma|$  for this  $r$ . Then  $\tau = Z \upharpoonright_n$  is a viable extension of  $\sigma$ , so  $\tau$ , or some prefix of it that is longer than  $\sigma$ , is in  $U$ .  $\square$

The Oberwolfach group (Bienvenu, Greenberg, Kucera, Nies, and Turetsky) [3, Cor. 5.5] showed that every OW-random is density random. The Madison group provided a direct proof of this fact. A left-c.e. bounded test is a nested sequence  $\langle \mathcal{V}_n \rangle$  of uniformly  $\Sigma_1^0$  classes such that for some computable sequence of rationals  $\langle \beta_n \rangle$  and  $\beta = \sup_n \beta_n$  we have  $\lambda(\mathcal{V}_n) \leq \beta - \beta_n$  for all  $n$ .  $Z$  fails this test if  $Z \in \bigcap_n \mathcal{V}_n$ . The OW group introduced this test notion and used it for one possible characterisation of OW randomness. The Madison group used these tests (formerly called Auckland tests) directly.

**Proposition 13.6.** *Every OW random  $Z$  is density random.*

*Proof.* Given left-c.e. martingale  $M$  we want to show that  $M$  converges along  $Z$ . Let  $M = \sup D_m$  where  $D_m$  is a computable rational valued martingale uniformly in  $m$ . Let  $\beta = M(\langle \rangle)$  and  $\beta_m = D_m(\langle \rangle)$  so that  $\beta = \sup_m \beta_m$ . Let  $L_m = M - D_m$  be the “rest” martingale at stage  $m$ .

Assume that  $M$  does not converge along  $Z$ . Multiplying  $M$  by a sufficiently large integer we may then assume that

$$1 < \limsup_k M(Z \upharpoonright_k) - \liminf_k M(Z \upharpoonright_k).$$

Define the left-c.e. bounded test by

$$\mathcal{V}_m = \{Y : \exists k L_m(Y \upharpoonright_k) > 1\}.$$

Then by the usual Kolmogorov inequality, we have  $\lambda \mathcal{V}_m \leq L_m(\langle \rangle) = \beta - \beta_m$ .

To show  $Z$  fails  $(\mathcal{V}_m)$ :  $Z$  is computably random, so  $l_m = \lim_k D_m(Z \upharpoonright_k)$  exists for each  $m$ . Furthermore,  $l_m \leq \inf_k M(Z \upharpoonright_k)$ . Thus  $\exists k L_m(Y \upharpoonright_k) > 1$ ; namely, the divergence is only due to the rest martingale at stage  $m$ .  $\square$

We note that this proof fails in the higher setting of randomness notions. See Section 14.

#### 14. NIES: DENSITY AND HIGHER RANDOMNESS

By Nies (August). The work of the Madison group described in Section 13 can be lifted to the domain of higher randomness. Interestingly, density one now can be equivalently required for any  $\Sigma_1^1$  class containing the real, not necessarily closed.

We use the following fact due to Greenberg. It is a higher analog of the original weaker version of Prop. 12.1.

**Proposition 14.1** (N. Greenberg, 2013). *Let  $\mathcal{C} \subseteq 2^{\mathbb{N}}$  be  $\Sigma_1^1$ . Let  $Z \in \mathcal{C}$  be  $\Pi_1^1$ -ML-random. Then  $\bar{\rho}_2(\mathcal{C} \mid Z) = 1$ .*

*Proof.* If  $\bar{\rho}_2(\mathcal{C} \mid Z) < 1$  then there is a positive rational  $q < 1$  and  $n^*$  such that for all  $n \geq n^*$  we have  $\lambda_{Z \upharpoonright_n}(\mathcal{C}) < q$ . Choose a rational  $r$  with  $q < r < 1$ . We define  $\Pi_1^1$ -anti chains in  $2^{<\omega}$   $U_n$ , uniformly in  $n$ . Let  $U_0 = \{\langle Z \upharpoonright_{n^*} \rangle\}$ . Suppose  $U_n$  has been defined. For each  $\sigma \in U_n$ , at a stage  $\alpha$  such that  $\lambda_\sigma(\mathcal{C}_\alpha) < q$ , we obtain effectively a hyper-arithmetical antichain  $V$  of extensions of  $\sigma$  such that  $\mathcal{C}_\alpha \cap [\sigma] \subseteq [V]^\prec$  and  $\lambda_\sigma([V]^\prec) < r$ . Put  $V$  into  $U_{n+1}$ .

Clearly  $\lambda U_n \leq r^n$  for each  $n$ . Also,  $Z \in \bigcap_n U_n$ , so  $Z$  is not  $\Pi_1^1$ -ML-random.  $\square$

A martingale  $M: 2^{<\omega} \rightarrow \mathbb{R}$  is called left- $\Pi_1^1$  if  $M(\sigma)$  is a left- $\Pi_1^1$  real uniformly in  $\sigma$ .

**Theorem 14.2.** *Let  $Z$  be  $\Pi_1^1$ -ML-random. The following are equivalent.*

- (i)  $\rho(\mathcal{C} \mid Z) = 1$  for each  $\Sigma_1^1$ -class  $\mathcal{C}$  containing  $Z$ .
- (ii)  $\rho(\mathcal{C} \mid Z) = 1$  for each closed  $\Sigma_1^1$ -class  $\mathcal{C}$  containing  $Z$ .
- (iii) each left- $\Pi_1^1$  martingale converges on  $Z$  to a finite value.

*Proof.* (iii)  $\rightarrow$  (i): The measure of a  $\Sigma_1^1$  set is left- $\Sigma_1^1$  in a uniform way (see e.g. [30, Ch. 9]). Therefore  $M(\sigma) = 1 - \lambda_\sigma(\mathcal{C})$  is a left- $\Pi_1^1$  martingale. Since  $M$  converges along  $Z$ , and since by Prop. 14.1  $\liminf_n M(Z \upharpoonright_n) = 0$ , it converges along  $Z$  to 0. This shows that  $\rho(\mathcal{C} \mid Z) = 1$ .

(ii)  $\rightarrow$  (iii). We follow the proof of the Madison Theorem 13.1 given above. All stages  $s$  are now interpreted as computable ordinals. Computable functions/ constructions, are now functions  $\omega_1^{CK} \rightarrow L_{\omega_1^{CK}}$  with  $\Sigma_1$  graph/ assignments of recursive ordinals to instructions.

**Definition 14.3.** A  $\Pi_1^1$ -Madison test is a  $\Sigma_1$  over  $L_{\omega_1^{CK}}$  function  $\langle U_s \rangle_{s < \omega_1^{CK}}$  mapping ordinals to (hyperarithmetical) subsets of  $2^{<\omega}$  such that  $U_0 = \emptyset$ , for each stage  $s$  we have  $\text{wt}(U_s) \leq c$  for some constant  $c$ , and for all strings  $\sigma, \tau$ ,

- (a)  $\tau \in U_s - U_{s+1} \rightarrow \exists \sigma \prec \tau [\sigma \in U_{s+1} - U_s]$
- (b)  $\text{wt}(\sigma^\prec \cap U_s) > 2^{-|\sigma|} \rightarrow \sigma \in U_s$ .

Also  $U_\gamma(\sigma) = \lim_{\alpha < \gamma} U_\alpha(\sigma)$  for each limit ordinal  $\gamma$ .

The following well-known fact can be proved similar to [30, 1.9.19].

**Lemma 14.4.** *Let  $\mathcal{A} \subseteq 2^\mathbb{N}$  be a hyperarithmetical open. Given a rational  $q$  with  $q > \lambda A$ , we can effectively determine from  $\mathcal{A}, q$  a hyperarithmetical open  $\mathcal{S} \supseteq \mathcal{A}$  with  $\lambda \mathcal{S} = q$ .*

**Lemma 14.5.** *Let  $Z$  be a  $\Pi_1^1$  ML-random such that  $\rho(\mathcal{C} \mid Z) = 1$  for each closed  $\Sigma_1^1$ -class  $\mathcal{C}$  containing  $Z$ . Then  $Z$  passes each  $\Pi_1^1$ -Madison test.*

The proof follows the proof of the analogous Lemma 13.3. The sets  $\mathcal{A}_{\sigma,s}^k$  are now hyperarithmetical open sets computed from  $k, \sigma, s$ . Suppose  $\sigma \in U_{s+1} - U_s$ . The set  $\tilde{\mathcal{A}}_{\sigma,s}^k$  is defined as before. To effectively obtain  $\mathcal{A}_{\sigma,s+1}^k$ , we apply Lemma 14.4 to add mass from  $[\sigma]$  to  $\tilde{\mathcal{A}}_{\sigma,s+1}^k$  in order to ensure  $\lambda(\mathcal{A}_{\sigma,s+1}^k) = 2^{-|\sigma|-k}$  as required.

As before let  $\mathcal{S}_t^k = \bigcup_{\sigma \in U_t} \mathcal{A}_{\sigma,t}^k$ . Then  $\mathcal{S}_t^k \subseteq \mathcal{S}_{t+1}^k$  by property (a) of  $\Pi_1^1$  Madison tests. Clearly  $\lambda \mathcal{S}_t^k \leq 2^{-k} \text{wt}(U_t) \leq 2^{-k}$ . So  $\mathcal{S}^k = \bigcup_{t < \omega_1^{CK}} \mathcal{S}_t^k$  determines a  $\Pi_1^1$  ML-test.

By construction  $\bar{\rho}(2^\mathbb{N} - \mathcal{S}^k \mid Z) \leq 1 - 2^{-k}$ . Since  $Z$  is ML-random we have  $Z \notin \mathcal{S}^k$  for some  $k$ . So  $\bar{\rho}(\mathcal{C} \mid Z) < 1$  for the closed  $\Sigma_1^1$ -class  $\mathcal{C} = 2^\mathbb{N} - \mathcal{S}^k$  containing  $Z$ .

The analog of Lemma 13.4 also holds.

**Lemma 14.6.** *Suppose that  $Z$  passes each  $\Pi_1^1$ -Madison test. Then every left- $\Pi_1^1$  martingale  $L$  converges along  $Z$ .*

The proof of 13.4 was already set up so that this works. The uniformly hyp labelling functions  $\gamma_s$  now map  $U_s$  to  $\omega_1^{CK}$ . Note that the antichains  $F$  can now be infinite.  $\square$

A  $\Pi_1^1$  ML-random satisfying any of the three conditions above will be called  $\Pi_1^1$ -density random. We note the following implications, none of which are known to be proper.

higher weak 2 random  $\Rightarrow \Pi_1^1$  OW-random  $\Rightarrow \Pi_1^1$  density random.

The first implication is due to Bienvenu, Greenberg and Monin. The second is the higher analog of Proposition 13.6:

**Proposition 14.7.** *Every  $\Pi_1^1$  OW random  $Z$  is  $\Pi_1^1$  density random.*

However, we need to go back to the original proof [3, Cor. 5.5]. The reason is that the left-c.e. bounded tests don't make sense in the higher setting; Oberwolfach tests, in contrast, can be suitably adapted. The  $n$ -th test component is obtained by counting  $n$  oscillations.

## 15. MIYABE: BEING A LEBESGUE POINT FOR EACH INTEGRAL TESTS

Input by Kenshi Miyabe.

**Theorem 15.1.** *The following are equivalent for a ML-random real  $z \in [0, 1]$ .*

- (1)  $z$  is a density-one point.
- (2) Every left-c.e. martingale converges on  $z$ .

A ML-random satisfying one of these conditions will be called *density random*. This theorem follows from the following theorems.

**Theorem 15.2** (Mushfeq Khan and Joseph Miller). *Let  $z$  be a ML-random dyadic density-one point. Then  $z$  is a full density-one point.*

**Theorem 15.3** (Bienvenu et al. [2]). *If every left-c.e. martingale converges on  $z$ , then  $z$  is a dyadic density-one point.*

**Theorem 15.4** (Andrews, Cai, Diamondstone, Lempp and Miller, 2012). *If  $z$  is a ML-random dyadic density-one point, then every left-c.e. martingale converges on  $z$ .*

Here, we give a characterization of density randomness via the Lebesgue differentiation theorem.

**Theorem 15.5.** *The following are equivalent for  $z \in [0, 1]$ :*

- (1)  *$z$  is density random.*
- (2)  *$z$  is a dyadic Lebesgue point for each integral test.*
- (3)  *$z$  is a Lebesgue point for each integral test.*

Recall that an *integral test* on  $[0, 1]$  with the Lebesgue measure is an integrable lower semicomputable function  $f : [0, 1] \rightarrow \overline{\mathbb{R}}^+$ .

Note that one direction is easy.

*Proof of (ii)  $\Rightarrow$  (i) of Theorem 15.5.* Suppose that  $z$  is a Lebesgue point for each integral test. Then  $f(z)$  is finite for each integral test  $f$ , whence  $z$  is ML-random.

Let  $C$  be a  $\Pi_1^0$  class containing  $z$ . We define a function  $f : [0, 1] \rightarrow \overline{\mathbb{R}}^+$  by

$$f(x) = \begin{cases} 1 & \text{if } x \notin C \\ 0 & \text{if } x \in C. \end{cases}$$

Then,  $f$  is an integral test. Since  $z$  is a Lebesgue point for  $f$ ,  $C$  has density-one at  $z$ .  $\square$

For the converse, we first show the following lemma.

**Lemma 15.6.** *If an ML-random set  $z$  is a dyadic weak Lebesgue point for an integral test  $f$ , then  $z$  is a dyadic Lebesgue point for  $f$ .*

*Proof.* As a notation, for a function  $f : \subseteq [0, 1] \rightarrow \mathbb{R}$  and  $z \in [0, 1]$ , let

$$D(f, \sigma) = \frac{\int_{[\sigma]} f \, d\mu}{2^{-n}}.$$

Then,  $z$  is a dyadic Lebesgue point iff  $\lim_n D(f, z \upharpoonright n) = f(z)$ . If  $f$  is a integral test, then  $D(f, -)$  is a left-c.e. martingale.

Suppose that  $z$  is not a dyadic Lebesgue point for an integral test  $f$  and  $z$  is a dyadic weak Lebesgue point for  $f$ . Then  $\lim_n D(f, z \upharpoonright n) =: r$  exists and  $f(z) \neq r$ .

Let

$$f = \sup_s f_s$$



where  $\{f_s\}$  is a computable sequence of rational step functions. Then, there is a computable order  $u$  such that  $D(f_s, \sigma) = D(f_s, \sigma 0) = D(f_s, \sigma 1)$  for each  $\sigma$  satisfying  $|\sigma| \geq u(s)$ . Unless  $z$  is a dyadic rational, we have

$$\lim_n D(f_s, z \upharpoonright n) = f_s(z).$$

Suppose that  $r < f(z)$ . Since  $\lim_s f_s(z) = f(z)$ , there is  $t$  such that

$$r < f_t(z) \leq f(z).$$

Then

$$r < f_t(z) = \lim_n D(f_t, z \upharpoonright n) \leq \lim_n D(f, z \upharpoonright n).$$

This is a contradiction.

Suppose that  $r > f(z)$ . Let  $q$  be a rational such that  $f(z) < q < r$ . We build a new integral test  $g$  such that  $g(z) = \infty$ .

We prepare auxiliary uniformly c.e. sets  $\{S_n\}$  where  $S_n \subseteq 2^{<\omega} \times \omega$  for each  $n$ . Let  $S_0 = \{(\lambda, 0)\}$  where  $\lambda$  is the empty string. For each  $n \geq 1$  and  $(\sigma, s) \in S_{n-1}$ , computably enumerate  $(\tau, t)$  into  $S_n^\sigma$  so that

- $\sigma \prec \tau$ ,
- $|\tau| \geq u(s)$ ,
- $D(f_t, \tau) > q$ ,
- $\{\tau \in 2^{<\omega} : (\tau, t) \in S_n^\sigma\}$  is prefix-free,

We can further assume that

$$\bigcup \{[\tau] : \sigma \prec \tau, |\tau| \geq u(s), D(f, \tau) > q\} = \bigcup \{[\tau] : (\tau, t) \in S_n^\sigma\}.$$

Let  $S_n = \bigcup_{(\sigma, s) \in S_{n-1}} S_n^\sigma$ .

For each  $(\tau, t) \in S_n$ , let

$$g_\tau = (q - D(f_s, \sigma)) \mathbf{1}_{[\tau]}$$

where  $(\sigma, s) \in S_{n-1}$  and  $\sigma \prec \tau$ . We define  $g$  by

$$g = \sum_n \sum_{(\tau, t) \in S_n} g_\tau.$$

Note that

$$\int g_\tau d\mu \leq (D(f_t, \tau) - D(f_s, \sigma)) 2^{-|\tau|} = \int_{[\tau]} (f_t - f_s) d\mu,$$

thus  $\int g d\mu \leq \int f d\mu < \infty$ . Hence,  $g$  is an integral test.

Since  $\lim_n D(f, z \upharpoonright n) = r > q$ , there exists  $(\tau_n, t_n) \in S_n$  such that  $\tau_n \prec z$  for each  $n$ . Then,

$$g(z) = \sum_n (q - D(f_s, \sigma)) \geq \sum_n (q - f(z)) = \infty.$$

□

*Proof of (i)  $\Rightarrow$  (ii) of Theorem 15.1.* Suppose that  $z$  is density random. Let  $f$  be an integral test. Then  $D(f, -)$  is a left-c.e. martingale. By Theorem 15.1,  $\lim_n D(f, z \upharpoonright n)$  exists, whence  $z$  is a dyadic weak Lebesgue point for  $f$ . By Lemma 15.6,  $z$  is a dyadic Lebesgue point for  $f$ . □

To drop “dyadic”, we recall the following results.

**Proposition 15.7.** *Let  $f : [0, 1] \rightarrow \mathbb{R}$  be interval-c.e. Then  $\tilde{D}_2 f(z) = \tilde{D}f(z)$  and  $\underline{D}_2 f(z) = \underline{D}f(z)$  for each non-porosity point  $z$ .*

**Lemma 15.8** ([7]; after Fact 2.4, Fact 7.2). *For each real  $z$ ,*

$$\underline{D}f(z) \leq \underline{D}_2 f(z) \leq \tilde{D}f(z) \leq \overline{D}f(z).$$

*If  $f$  is continuous, then*

$$\underline{D}f(z) = \underline{D}_2 f(z) \text{ and } \tilde{D}f(z) = \overline{D}f(z).$$

**Lemma 15.9** (Lemma 3.8 in [5]). *Let  $C$  be a  $\Pi_1^0$  class. If  $z \in C$  is difference random, then  $C$  is not porous at  $z$ .*

*Proof of (ii)  $\iff$  (iii) of Theorem 15.1.* Note that (iii)  $\Rightarrow$  (ii) holds by definition.

We prove (ii)  $\Rightarrow$  (iii). Suppose that  $z$  is a dyadic Lebesgue point for each integral test. Then,  $z$  is density random, whence difference random, thus a non-porosity point.

Let  $f$  be an integral test. Then,  $F(x) = \int_{[0,x]} f \, d\mu$  is interval-c.e. and continuous. Hence,

$$\limsup_{Q \rightarrow z} \frac{\int_Q f \, d\mu}{\mu(Q)} = \overline{D}F(z) = \tilde{D}F(z) = \tilde{D}_2 F(z) = \limsup_{n \rightarrow \infty} \frac{\int_{[z \upharpoonright n]} f \, d\mu}{\mu(Q)} = f(z).$$

Similarly, we have  $\liminf_{Q \rightarrow z} \frac{\int_Q f \, d\mu}{\mu(Q)} = f(z)$ , whence  $z$  is a Lebesgue point for  $f$ .  $\square$

Actually, only one integral test characterizes density randomness.

**Lemma 15.10.** *Let  $f, g$  be integral tests. If an ML-random set  $x$  is a dyadic weak Lebesgue point for  $f + g$ , then  $x$  is a dyadic weak Lebesgue point for  $f$ .*

*Proof.* Suppose that  $x$  is a dyadic weak Lebesgue point for  $f + g$  and  $x$  is not a dyadic Lebesgue point for  $f$ . Let

$$r = \lim_n D(f + g, x \upharpoonright n).$$

Then, there are rationals  $p, q$  ( $p < q$ ) such that  $D_n(f, x \upharpoonright n) > q$  for infinitely many  $n$  and  $D_n(f, x \upharpoonright n) < p$  for infinitely many  $n$ . Notice that  $q \leq r$ . By replacing  $q$  with  $\frac{p+q}{2}$ , we can assume that  $q < r$ . Let  $\epsilon = \frac{q-p}{3} > 0$ . Then, there is a natural number  $N$  such that, for each  $n > N$ , we have

$$|D(f + g, x \upharpoonright n) - r| < \epsilon.$$

Hence,

$$r - \epsilon < D(f, x \upharpoonright n) + D(g, x \upharpoonright n) < r + \epsilon.$$

If  $D(f, x \upharpoonright n) > q$ , then

$$D(g, x \upharpoonright n) < r + \epsilon - q.$$

If  $D(f, x \upharpoonright n) < p$ , then

$$D(g, x \upharpoonright n) > r - \epsilon - p.$$

If  $D(g, x \upharpoonright n) > r - \epsilon - p$ , then

$$D(f, x \upharpoonright n) < r + \epsilon - r + \epsilon + p = 2\epsilon + p < q.$$

We consider the following betting strategy. First use the strategy  $f$  until  $D(f, x \upharpoonright n) > q$ . When found, stop betting until  $D(g, x \upharpoonright n) > r - \epsilon - q$ . At the stage  $n$ , use the strategy

$$\frac{q}{2\epsilon + p}f.$$

Then,  $x$  is not ML-random. □

**Theorem 15.11.** *Let  $f$  be a Solovay-complete integral test. Then  $x$  is a Lebesgue point for  $f$  if and only if  $x$  is density random.*

*Proof.* The “if” direction follows from Theorem 15.5.

Suppose  $x$  is not density random. We can assume that  $x$  is ML-random, because, otherwise,  $f(x) = \infty$  and  $x$  is not a dyadic Lebesgue point for  $f$ . Then there is an integral test  $g$  such that  $x$  is not a dyadic Lebesgue point for  $g$ . Since  $f$  is Solovay-complete, there are a rational  $q$  and an integral test  $h$  such that

$$f = \frac{g}{q} + h.$$

Notice that  $x$  is not a dyadic Lebesgue point for  $\frac{g}{q}$ . By Lemma 15.6,  $x$  is not a dyadic weak Lebesgue point for  $\frac{g}{q}$ . By the lemmas above,  $x$  is not a dyadic weak Lebesgue point for  $f$ . Thus,  $x$  is not a Lebesgue point for  $f$ . □

## Part 4. Similarity relations for Polish metric spaces

In October 2013, André Nies gave a talk as part of the Universality and Homogeneity Trimester at the Hausdorff Institute for Mathematics (HIM) in Bonn. The summary follows.

We are given a class of structures. We always mean concrete presentations of structures (rather than “up to isomorphism”). We address the following **leading questions** for this class:

- (a) Which similarity relations are there on the class?
- (b) How complex are these similarity relations?
- (c) If structures  $X, Y$  in the class are similar, how complex, relative to  $X, Y$ , is the means for showing this? For instance, if  $X \cong Y$ , can one compute an isomorphism from the structures?

In the model theoretic setting, we could be given the countable models of a first-order theory. In this setting, some answers to these questions are:

- (a) isomorphism  $\cong$ , elementary equivalence  $\equiv$ , elementary equivalence  $\equiv_\alpha$  for  $L_{\omega_1, \omega}$  sentences of rank  $< \alpha$ .
- (b) Isomorphism of countable graphs, linear orders, countable Boolean algebras is  $\leq_B$  complete for orbit equivalence relations of continuous  $S_\infty$  actions (where  $\leq_B$  is Borel reducibility, and  $S_\infty$  is the Polish group of permutations of  $\omega$ ).
- (c) Suppose the similarity is  $\cong$ . For certain natural classes, this question has been answered in computable model theory. That area introduced the notion of being *relatively computably categorical*, where presentations of  $X, Y$  together uniformly compute an isomorphism if there is one at all. For instance, a dense linear order is r.c.c. There are variants, such as being *uniformly computably categorical*, where one computes an isomorphism from computable indices for the structures.

We will be mainly considering the **metric** setting. We are given a class of Polish metric spaces. To answer (a): The following similarities, which will be defined formally below, have been studied.

Isometry  $\cong_i$ , homeomorphism  $\cong_h$ ,

Gromov-Hausdorff distance 0, Lipschitz equivalence.

The former two are discussed in detail in [18, Ch. 14]. The latter two are due to Gromov; see his book [23, Ch.3] (the first edition dates from 1998). After some preliminary facts, we will answer (b) and (c) for the metric setting. We also consider Polish metric spaces with some additional structure, such as Banach spaces, or spaces with a probability measure on the Borel sets.

## 16. REPRESENTING POLISH METRIC SPACES

We adopt the global view. Single structures are thought of as points in a “hyperspace”. To endow this hyperspace with its own structure, it matters how we represent a single structure. For metric spaces, two ways are common.

- (1) Let  $\mathbb{U}$  denote the Urysohn space. Let  $F(\mathbb{U})$  denotes its Effros algebra, which is a  $\sigma$ -algebra where the points are closed subsets of  $\mathbb{U}$ . Each Polish metric space is isometric to an element of  $F(\mathbb{U})$ . See Gao [18, Ch. 14].
- (2) A point  $V = \langle v_{i,k} \rangle_{i,k \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N} \times \mathbb{N}}$  is a *distance matrix* if  $V$  is a pseudo-metric on  $\mathbb{N}$ . Let  $M_V$  denote the completion of the corresponding pseudo-metric space. This means that in  $M_V$  we have a distinguished dense sequence of points  $\langle p_i \rangle$  and present the space by giving their distances. We merely ask that  $V$  is a pseudo-metric in order to ensure that the set  $\mathcal{M}$  of distance matrices is closed in  $\mathbb{R}^{\mathbb{N} \times \mathbb{N}}$ .

Both representations are in a sense equivalent as pointed out for instance in [18, Ch. 14]. However, the second one is better for studying the complexity of the space. For instance, a computable metric space  $(M, d, \langle p_i \rangle)$  is given by a distance matrix  $w$  such that  $w_{i,k} = d(p_i, p_k)$  is a computable real uniformly in  $i, k$ .

A Polish group action is a continuous action  $G \times X \rightarrow X$  where  $G$  is a Polish group and  $X$  a Polish space. We write  $G \curvearrowright X$  to say that  $G$  acts on  $X$  continuously. The corresponding *orbit equivalence relation* is  $E_G^X = \{ \langle x, y \rangle : \exists g [gx = y] \}$ .

## 17. POLISH METRIC SPACES AND THE CLASSICAL SCOTT ANALYSIS.

A metric space  $(M, d)$  can be turned into a structure in the language with binary relations  $S_q$  for  $q \in \mathbb{Q}^+$ , where  $S_q(a, b)$  holds if  $d(a, b) < q$ .

**Definition 17.1.** Let  $M$  be an  $\mathcal{L}$ -structure. We define inductively what it means for finite tuples  $\bar{a}, \bar{b}$  from  $M$  of the same length to be  $\alpha$ -equivalent, denoted by  $\bar{a} \equiv_\alpha \bar{b}$ .

- $\bar{a} \equiv_0 \bar{b}$  if and only if the quantifier-free types of the tuples are the same.
- For a limit ordinal  $\alpha$ ,  $\bar{a} \equiv_\alpha \bar{b}$  if and only if  $\bar{a} \equiv_\beta \bar{b}$  for all  $\beta < \alpha$ .
- $\bar{a} \equiv_{\alpha+1} \bar{b}$  if and only if both of the following hold:
  - For all  $x \in M$ , there is some  $y \in M$  such that  $\bar{a}x \equiv_\alpha \bar{b}y$
  - For all  $y \in M$ , there is some  $x \in M$  such that  $\bar{a}x \equiv_\alpha \bar{b}y$

The *Scott rank*  $\text{sr}(M)$  of a structure  $M$  is defined as the smallest  $\alpha$  such that  $\equiv_\alpha$  implies  $\equiv_{\alpha+1}$  for all tuples of that structure. We remark that always  $\text{sr}(M) < |M|^+$ .

**Fact 17.2.** A Polish space has Scott rank 0 iff it is ultrahomogeneous.

Friedman, Körwien and Nies (2012) have shown that for each  $\alpha < \omega_1$ , there is a countable Polish ultrametric space  $M$  such that  $\text{sr}(M) = \alpha \times \omega$ .

### Question 17.3.

- (a) Does every Polish metric space have countable Scott rank?
- (b) Can it in fact be described within the class of Polish metric spaces by an  $L_{\omega_1, \omega}$  sentence?

Note (Feb 2014). Question (a) has been answered in the affirmative by Michal Ducha, a postdoc from Warsaw (student of J. Zapletal) who participated in the HIM program.

18. ISOMETRY  $\cong_i$ 

In 1998 Anatoly Vershik [39] asked about the complexity of isometry  $\cong_i$  on Polish metric spaces, and in particular if one can assign invariants. The answer was a resounding no. By the following result,  $\cong_i$  is Borel equivalent to  $E_{\text{Iso}(\mathbb{U})}^{F(\mathbb{U})}$ , the universal orbit equivalence relation given by the action of the isometry group of  $\mathbb{U}$  on the Effros algebra of  $\mathbb{U}$ .

**Theorem 18.1** (Gao-Kechris 2000; Clemens; see [18], Ch. 14).

- (1)  $\cong_i \leq_B E_{\text{Iso}(\mathbb{U})}^{F(\mathbb{U})}$ .
- (2) For every Polish group action  $G \curvearrowright X$  we have  $E_G^X \leq_B \cong_i$ .

Let  $\mathcal{K}$  be the class of compact metric spaces. Note that this is  $\Pi_3^0$  with respect to the distance matrix representation of Polish metric spaces, because compactness is equivalent to being totally bounded. Isometry of compact spaces is much simpler than in the general case: the points in some fixed Polish space can serve as invariants.

**Theorem 18.2** (Essentially Gromov [23], Thm 3.27.5).

$$\cong_i \cap (\mathcal{K} \times \mathcal{K}) \leq_B \text{id}_{\mathbb{R}}.$$

*Proof.* Gromov shows that the sequence of sets of  $n \times n$  distance matrices that occur in a compact space  $X$  constitute a complete set of invariants. Each such matrix is a point in a compact set  $K_n(X) \subseteq \mathbb{R}^{n^2}$ . The sequence of such compact sets can be represented by a single point in a Polish space, say  $\mathbb{R}$ .  $\square$

*Computable versions.* The distance matrices  $V = \langle v_{i,k} \rangle_{i,k \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N} \times \mathbb{N}}$  form an effectively closed set. They can in fact be coded as the infinite branches of a  $\Pi_1^0$  tree  $\subseteq 2^{<\omega}$ . Such a branch provides yes/no answers to queries of the form  $|v_{i,k} - q| < \epsilon$  for  $i, k \in \mathbb{N}$ ,  $q \in \mathbb{Q}_0^+$ , and  $\epsilon \in \mathbb{Q}^+$ .

Let  $V_e$  denote the  $e$ -th partial computable distance matrix. The domain of this partial computable function grows as long as the data are consistent with being a distance matrix; if they are seen to be not (a  $\Sigma_1^0$  event) it stops, so that the function is only defined on an initial segment of  $\mathbb{N}$ . Being total is  $\Pi_2^0$ .

Let  $M_e$  denote the computable metric space given by the  $e$ -th (total) distance matrix  $V_e$ . The following can be proved by computably reducing the isomorphism problem for computable graphs by Fokina et al. [15].

**Proposition 18.3.**  $\{\langle e, k \rangle : M_e \cong_i M_k\}$  is complete for  $\Sigma_1^1$  equivalence relations on  $\omega$  with respect to computable reductions.

**Proposition 18.4** (Melnikov and Nies [28]). The set  $C$  of indices for compact computable metric spaces is  $\Pi_3^0$ . Isometry is  $\Pi_2^0$  within that set, that is, of the form  $E \cap C \times C$  where  $E$  is a  $\Pi_2^0$  relation.

## 19. II. HAVING GROMOV-HAUSDORFF DISTANCE 0.

The following is ongoing work of Itai Ben Yaacov, Nies, and Todor Tsankov. One thinks of two metric spaces  $X, Y$  as isometric within error  $\epsilon$  if they can be isometrically embedded into a third metric space  $Z$  in such a way that

the usual Hausdorff distance of the two images is at most  $\epsilon$ . “ $X, Y$  isometric within error 0” clearly means that the completions of  $X, Y$  are isometric. The Gromov- Hausdorff distance of  $X, Y$  is defined by

$$d_{GH}(X, Y) = \inf\{\epsilon: X, Y \text{ are isometric within error } \epsilon\}.$$

(Also see Subsection 19.2 for an equivalent definition.)

For instance, if we let  $X = \{0, 1\}$  and  $Y = \{1/4, 3/4\}$ , then

$$d_{GH}(X, Y) = 1/4.$$

So, are there examples of non-isometric spaces  $X, Y$  with GH-distance 0? If so, neither  $X$  nor  $Y$  can be compact (Gromov). Also there is no positive lower bound on the distance of distinct points, otherwise a near isometry with error less than that bound will be an isometry. During the HIM talk, Nies mentioned an example: let  $\mathbb{E}$  be the unit sphere of the Gurarij space. Let  $v \in \mathbb{E}$  be smooth, and  $w$  be non-smooth. Let  $X = Y = \mathbb{E} \cup \{a, b\}$ , with  $d_X(a, b) = d_Y(a, b) = 3$ . We set  $d_X(v, a) = d_X(v, b) = 3$ , and  $d_Y(w, a) = d_Y(w, b) = 3$ . Any isometry would have to map  $v$  to  $w$ , which is impossible. However, by general properties of the Gurarij space,  $d_{GH}(X, Y) = 0$ .

**19.1. Fact and more examples for GH-distance.** After Nies’ HIM talk, Matatiahou Rubin and Philipp Schlicht constructed further, simpler examples. Let  $B_X$  denote the unit ball of a Banach space  $X$ .

**Proposition 19.1.** *There are nonisometric Banach spaces  $X, Y$  with  $d_{GH}(B_X, B_Y) = 0$ .*

To prove this let  $D = \langle p_i \rangle_{i \in \mathbb{N}}$  be a dense sequence of distinct elements in  $(1, 2)$ , say. Let  $U_p$  be the 2-dimensional  $\mathbb{R}$  vector space with  $\ell_p$  norm. Let  $E_D$  be the  $c_0$ -sum of the spaces  $U_{p_i}$ . That is, null sequences, with norm the maximum of the individual  $\ell_{p_i}$  norms. If we have two dense sequences with different sets of members, the unit balls of the spaces are at distance 0, but not isometric.

Let  $\mathbb{G}$  denote the Gurarij space. By a (continuous) model-theoretic argument, related to  $\aleph_0$ -categoricity, one can show that if  $X$  is a Banach space and  $d_{GH}(B_X, B_{\mathbb{G}}) = 0$ , then  $X$  is isometric to  $\mathbb{G}$ .

**Remark 19.2** (Melleray-Schlicht).

- (1) *Any two separable Banach spaces  $X, Y$  with  $d_{GH}(X, Y) < \infty$  are isometric.*
- (2) *Isometry on Polish spaces reduces to  $E_{GH}$ .*

*Proof.* For Banach spaces  $X, Y$ ,  $(X, Y) \in E_{GH}$  and  $(X, Y) \in E_{GH}^\infty$  are equivalent by rescaling (i.e. rescaling the metric space into which we embed the spaces). It follows from the Main Theorem in a paper by Omladic and Semrl [33] that it is sufficient to prove that there is an  $\epsilon$ -isometry  $T: X \rightarrow Y$  for some  $\epsilon$ . For any two perfect Polish spaces with  $d_{GH}(X, Y) = 0$ , we can construct a bijective  $\epsilon$ -isometry by a straightforward back-and forth argument.

The second claim follows from a paper of Melleray [27], where he shows that isometry of Polish spaces reduces to isometry of Banach spaces.  $\square$

The following result of Schlicht and Rubin shows that there is a single  $E_{GH}$  class such that the isometry equivalence relation inside is Borel bi-reducible

with identity on  $2^{\mathbb{N}}$ . In particular, there are continuum many non-isometric spaces with discrete topology that are mutually at GH distance 0.

We equip  $[0, \epsilon] \times (\omega + 1) \times \mathbb{R}$  with the metric  $d$  defined by

$$\begin{aligned} d((x, i, y), (x', i', y')) &= 1 \text{ if } (x, i) \neq (x', i') \text{ and} \\ d((x, i, y), (x', i', y')) &= |y - y'| \text{ if } (x, i) = (x', i'). \end{aligned}$$

**Definition 19.3.** Suppose that  $f: [0, \epsilon] \rightarrow \omega + 1$  is a function.

- (1) Let  $X_f = \{(x, i, 0), (x, i, x) \in [0, \epsilon] \times (\omega + 1) \times \mathbb{R} \mid i \leq f(x)\}$  with the metric from  $[0, \epsilon] \times (\omega + 1) \times \mathbb{R}$ .
- (2) Let  $\text{supp}(f) = \{x \in [0, \epsilon] \mid f(x) \neq 0\}$  denote the *support* of  $f$ .
- (3) Let  $\text{bound}(f) = \{(x, i) \mid x \in \text{supp}(f), i \leq f(x)\}$ .

If  $|\text{supp}(f)| = \omega$ , then  $X_f$  is a discrete countable metric space with distance set  $\text{supp}(f) \cup \{0, 1\}$ .

**Proposition 19.4.** Suppose that  $\epsilon \leq \frac{1}{2}$ . Suppose that  $f_0: [0, \epsilon] \rightarrow \omega + 1$  is a function such that  $\text{supp}(f_0)$  is a countable dense subset of  $[0, \epsilon]$ . Then  $\text{id}_{\omega_2}$  is Borel reducible to  $\text{Iso} \upharpoonright [X_{f_0}]_{GH}$ .

*Proof.* Note that for arbitrary functions  $f, g: [0, \epsilon] \rightarrow \omega + 1$ ,  $X_f, X_g$  are isometric if and only if  $f = g$ .

**Claim 19.5.** Suppose that  $f, g: [0, \epsilon] \rightarrow \omega + 1$  are functions such that  $\text{supp}(f), \text{supp}(g)$  are countable dense subsets of  $[0, \epsilon]$ . Then  $d_{GH}(X_f, X_g) = 0$ .

*Proof.* Note that for every  $\delta > 0$ , there is a bijection  $h: \text{bound}(f) \rightarrow \text{bound}(g)$  such that  $|x - h(x, i)_0| < \delta$  for all  $(x, i) \in \text{bound}(f)$ . Let  $h \times \text{id}: X_f \rightarrow \text{bound}(g) \times \mathbb{R}$ ,  $(h \times \text{id})(x, i, y) = (h(x, i), y)$ . Then  $h \times \text{id}$  is distance preserving and  $d_H((h \times \text{id})[X_f], X_g) \leq \delta$ . Hence  $d_{GH}(X_f, X_g) \leq \delta$ .  $\square$

Let  $D_q = \{0, q\}$  for  $q > 0$ . Suppose that  $(q_n, i_n)_{n \in \omega}$  is an enumeration of  $\text{bound}(f_0)$  without repetitions. Suppose that  $X$  is a complete metric space with  $d_{GH}(X, X_{f_0}) = 0$ . Suppose that  $0 < \delta < 1$ . Since  $d_{GH}(X, X_{f_0}) < \frac{\delta}{3}$ ,  $X$  is of the form  $X = \bigsqcup_{n \in \omega} X_n^\delta$  with

- (1)  $d_{GH}(X_n^\delta, D_{q_n}) < \delta$  and
- (2)  $|d(x, y) - 1| < \delta$  if  $x \in X_m^\delta, y \in X_n^\delta$ , and  $m \neq n$ .

Let  $X_n = X_n^{\frac{1}{2}}$ . Conditions 1 and 2 imply that for all  $\delta < \frac{1}{2}$  and all  $n$ , there is some  $m$  with  $X_m^\delta = X_n$ . Hence for each  $n$  there is a sequence  $(n_i)_{i \in \omega}$  in  $\omega$  with  $d_{GH}(X_n, D_{q_{n_i}}) < \frac{1}{2^i}$ . It follows that  $1 \leq |X_n| \leq 2$ . Let  $p_n = d(x, y)$  if  $X_n = \{x, y\}$ . Let  $A = \{p_n \mid n \in \omega\}$ .

**Claim 19.6.**  $d(x, y) = 1$  for all  $x \in X_m$  and  $y \in X_n$  with  $m \neq n$ .

*Proof.* This follows from Condition 2 and since for all  $\delta < \frac{1}{2}$  and all  $k$ , there is some  $l$  with  $X_l^\delta = X_k$ .  $\square$

**Claim 19.7.**  $A \subseteq [0, \epsilon]$ .

*Proof.* Suppose that  $X_n = \{x, y\}$  and  $\eta = d(x, y) - \epsilon > 0$ . Suppose that  $X_n = X_m^\eta$ . This contradicts the fact that  $d_{GH}(X_m^\eta, D_{q_m}) < \eta$  by Condition (1).  $\square$



**Claim 19.8.** *A is dense in  $(0, \epsilon)$ .*

*Proof.* Suppose that  $U \subseteq (0, \epsilon)$  is nonempty and open with  $U \cap A = \emptyset$ . Suppose that  $(q_n - \delta, q_n + \delta) \subseteq U$ . This contradicts the fact that  $d_{GH}(X_n^{\frac{\delta}{2}}, D_{q_n}) < \frac{\delta}{2}$  by Condition (1).  $\square$

Let  $f: [0, \epsilon] \rightarrow \omega + 1$ ,  $f(x) = 0$  if  $x \notin A$ ,  $f(0) = i$  if  $|\{n \in \omega \mid |X_n| = 1\}| = i$ , and  $f(z) = i$  if  $|\{n \in \omega \mid \exists x, y \in X_n \ d(x, y) = z\}| = i$  for  $z \in (0, \epsilon]$ . Then  $X_f, X$  are isometric.  $\square$

**19.2. Bi-Katetov functions.** One can describe being isometric within error  $\epsilon$  without referring to a third space. A *bi-Katetov function*  $f: X \times Y \rightarrow \mathbb{R}$  is defined as

$$f(x, y) = d_Z(i(x), j(y)),$$

where  $i, j$  are embeddings into some metric space as above. Equivalently,  $f$  is 1-Lipschitz in both variables and

$$\begin{aligned} d_A(x, w) &\leq f(x, y) + f(w, y) \\ d_B(y, z) &\leq f(x, y) + f(x, z) \end{aligned}$$

A bi-Katetov function  $f$  can be seen as an approximate isometry. Its error  $q_f$  is given by

$$q_f = \max(\sup_x \inf_y f(x, y), \sup_y \inf_x f(x, y)).$$

By definition this equals the Hausdorff distance of the isometric images above.

For instance, if there is an actual onto isometry  $\theta: X \rightarrow Y$ , we can let  $f(x, y) = d_Y(\theta(x), y)$  and obtain the least possible error 0. Conversely, as mentioned above, if the spaces are complete and the error is 0 then there is an onto isometry.

Clearly we have

$$d_{GH}(X, Y) = \inf_f q_f,$$

where  $f$  runs through all the bi-Katetov functions on  $X \times Y$ .

**Remark 19.9.** *f A  $\subseteq X$  and  $B \subseteq Y$ , then any bi-Katetov function defined on  $A \times B$  extends to one  $f'$  defined on  $X \times Y$ . One uses amalgamation:*

$$f'(x, y) = \inf_{a \in A, b \in B} d_X(x, a) + f(a, b) + d_Y(b, y).$$

**19.3. Continuous Scott analysis.** We define approximations to  $d_{GH}$  from below by induction on countable ordinals.

Suppose  $\bar{a} = \langle a_i \rangle_{i < n}$  and  $\bar{b} = \langle b_i \rangle_{i < n}$  are enumerated finite metric spaces. Following Uspenskii [38] define

$$r_{0,n}(\bar{a}, \bar{b}) = \inf_{f \text{ is bi-Katetov on } \bar{a} \times \bar{b}} \max_{i < n} f(a_i, b_i).$$

Uspenskii gives an explicit expression for this in [38, Proposition 7.1]:

$$(4) \quad r_{0,n}(\bar{a}, \bar{b}) = \varepsilon/2 \text{ where } \varepsilon = \max_{i, k < n} |d(a_i, a_k) - d(b_i, b_k)|.$$

(In fact, Uspenskii builds a bi-Katetov function such that  $f(a_i, b_i) = \varepsilon/2$  for each  $i$ .)

**Definition 19.10.** Suppose  $A$  and  $B$  are metric spaces and  $\bar{a} \in A^n, \bar{b} \in B^n$ . Define by induction on ordinals  $\alpha$ :

$$\begin{aligned} r_{0,n}^{A,B}(\bar{a}, \bar{b}) &= r_{0,n}(\bar{a}, \bar{b}) \\ r_{\alpha+1,n}^{A,B}(\bar{a}, \bar{b}) &= \max \left( \sup_{x \in A} \inf_{y \in B} r_{\alpha,n+1}^{A,B}(\bar{a}x, \bar{b}y), \sup_{y \in B} \inf_{x \in A} r_{\alpha,n+1}^{A,B}(\bar{a}x, \bar{b}y) \right) \\ r_{\alpha,n}^{A,B}(\bar{a}, \bar{b}) &= \sup_{\beta < \alpha} r_{\beta,n}^{A,B}(\bar{a}, \bar{b}), \quad \text{for } \alpha \text{ a limit ordinal.} \end{aligned}$$

Given a metric space  $(X, d)$  and  $n \geq 1$ , we equip  $X^n$  with the “maximum” metric  $d(\bar{u}, \bar{v}) = \max_{i < n} d(u_i, v_i)$ . The following are not hard to check.

**Lemma 19.11.** *Fix separable metric spaces  $A, B$  of finite diameter.*

- (1) *For each  $\alpha$  and each  $n$ , the functions  $r_{\alpha,n}^{A,B}(\bar{a}, \bar{b})$  are 1-Lipschitz in  $\bar{a}$  and  $\bar{b}$ .*
- (2) *The functions  $r_{\alpha,n}^{A,B}(\bar{a}, \bar{b})$  are nondecreasing in  $\alpha$ .*
- (3) *There is  $\alpha < \omega_1$  after which all the  $r_{\alpha,n}^{A,B}$  stabilize.*

**Theorem 19.12** (Ben Yaacov, Nies, Tsankov 2013). *Let  $A, B$  be separable metric spaces of finite diameter. Let  $\alpha^*$  be such that  $r_{\alpha^*+1,n}^{A,B} = r_{\alpha^*,n}^{A,B}$  for each  $n$ . Then*

$$r_{\alpha^*,0}^{A,B} = d_{GH}(A, B).$$

*Proof.* Since  $A, B$  are fixed we suppress them in our notations. Variables  $a, a_i$  etc range over  $A$ , and  $b_i$  etc. range over  $B$ . For tuples  $\bar{a}, \bar{b}$  of length  $k$ , let

$$\delta_k(\bar{a}, \bar{b}) = \inf_f \{ \max(q_f, \max_{i < k} f(a_i, b_i)) \},$$

where  $f$  ranges over bi-Katetov functions on  $A \times B$ . We show that for each  $n$  and tuples  $\bar{a}, \bar{b}$  of length  $n$ ,

$$r_{\alpha^*,n}(\bar{a}, \bar{b}) = \delta_n(\bar{a}, \bar{b}).$$

For  $n = 0$  this establishes the theorem.

Firstly, we show by induction on ordinals  $\alpha$  that

$$r_{\alpha,n}(\bar{a}, \bar{b}) \leq \delta_n(\bar{a}, \bar{b}).$$

The cases  $\alpha = 0$  and  $\alpha$  limit ordinal are immediate. For the successor case, suppose that  $\delta_n(\bar{a}, \bar{b}) < s$  via a bi-Katetov function  $f$  on  $A \times B$ . For each  $x \in A$  we can pick  $y \in B$  such that  $f(x, y) < s$ . Then  $\delta_{n+1}(\bar{a}x, \bar{b}y) < s$  via the same  $f$ . Inductively we have  $r_{\alpha,n+1}(\bar{a}x, \bar{b}y) < s$ . Similarly, for each  $y \in B$  we can pick  $x \in A$  such that  $r_{\alpha,n+1}(\bar{a}x, \bar{b}y) < s$ . This shows that  $r_{\alpha+1,n}(\bar{a}, \bar{b}) \leq s$ .

Secondly, we verify that

$$\delta_n(\bar{a}, \bar{b}) \leq r_{\alpha^*,n}(\bar{a}, \bar{b})$$

Let  $r_{\alpha^*,n}(\bar{a}, \bar{b}) < t$ . We combine a back-and-forth argument with the compactness of the space of bi-Katetov functions in order to build a bi-Katetov function  $f$  with  $q_f \leq t$  and  $\max_{i < n} f(a_i, b_i) \leq t$ .

To do so we extend  $\bar{a}, \bar{b}$  to dense sequences in  $A, B$  respectively. Let  $D \subseteq A, E \subseteq B$  be countable dense sets. Let  $\bar{u}^k$  denote a tuple of length  $k$ ; in particular, we can write  $\bar{a} = \bar{a}^n$  and  $\bar{b} = \bar{b}^n$ . We ensure that

$$r_{\alpha^*, k}(\bar{a}^k, \bar{b}^k) < t \text{ for each } k \geq n.$$

Suppose  $\bar{a}^k, \bar{b}^k$  have been defined. If  $k$  is even, let  $a_k$  be the next element in  $D$ . Using  $r_{\alpha^*+1, k}(\bar{a}^k, \bar{b}^k) = r_{\alpha^*, k}(\bar{a}^k, \bar{b}^k)$  we can choose  $b_k$  so that  $r_{\alpha^*, k+1}(\bar{a}^{k+1}, \bar{b}^{k+1}) < t$ . Similarly, if  $k$  is odd, let  $b_k$  be the next element in  $E$  and choose  $a_k$  as required.

By Lemma 19.11(2) we have  $r_{0, k}(\bar{a}^k, \bar{b}^k) < t$  for each  $k \geq n$  via some bi-Katetov function  $\tilde{f}_k$  defined on  $\{a_0, \dots, a_{k-1}\} \times \{b_0, \dots, b_{k-1}\}$ . By Remark 19.9 we can extend this to a bi-Katetov function  $f_k$  defined on  $A \times B$ . By the compactness of the space of bi-Katetov functions on  $A \times B$ , viewed as elements of  $\mathbb{R}^{D \times E}$ , there is a subsequence  $k_0 < k_1 < \dots$  such that  $\langle f_{k_u} \rangle$  converges pointwise to a bi-Katetov function  $f$ . Since bi-Katetov functions are 1-Lipschitz in both arguments, this implies  $\lim_u f_{k_u}(a_p, b_p) = f(a_p, b_p)$  for each  $p$ . Therefore  $f(a_p, b_p) \leq t$ . This implies  $q_f \leq t$  as required.  $\square$

**Definition 19.13.** The *continuous Scott rank* of  $A$  is the least  $\alpha$  for which

$$r_{\alpha, n}^{A, A}(\bar{a}_1, \bar{a}_2) = r_{\alpha+1, n}^{A, A}(\bar{a}_1, \bar{a}_2), \quad \text{for all } n, \bar{a}_1, \bar{a}_2 \in A^n.$$

One can define an equivalence relation  $E_{GH}$  on the set of distance matrices  $\mathcal{M}$  by

$$AE_{GH}B \iff d_{GH}(A, B) = 0.$$

Using the continuous Scott analysis we can show:

**Theorem 19.14.** *Each equivalence class of  $E_{GH}$  is Borel.*

*Proof.* By induction each  $r_{\alpha, n}$  is a Borel function  $\mathcal{M} \times \mathcal{M} \times \mathbb{N}^n \times \mathbb{N}^n \rightarrow [0, 1]$ . Next one needs to prove the following. Fix  $A_0 \in \mathcal{M}$  and let  $\alpha_0 = \text{rank} A_0$ .

- for each  $\alpha$ , the set  $\{B \in \mathcal{M} : \text{rank}(B) = \alpha\}$  is Borel;
- $BE_{GH}A_0 \iff \text{rank}(B) = \alpha_0 \wedge r_{\alpha_0, 0}^{A_0, B} = 0$ .

$\square$

**Question 19.15.** *Is the function  $d_{GH}(A_0, \cdot)$  Borel on  $\mathcal{M}$ ?*

## 20. III. HOMEOMORPHISM $\cong_h$

We collect some results, most of which are proved in [18, Ch. 14]. For general Polish metric spaces,  $\cong_h$  is merely known to be  $\Sigma_2^1$ . Homeomorphism of compact metric spaces  $X, Y$  is analytic, because homeomorphisms are uniformly continuous. In fact, by the Banach-Stone theorem, we have

$$X \cong_h Y \iff \mathcal{C}(X) \cong_i \mathcal{C}(Y);$$

so by the aforementioned results of Gao and Kechris on isometry [19],  $\cong_h$  on compact metric spaces is Borel reducible to an orbit equivalence relation. (A similar argument works for locally compact metric spaces, using  $\mathcal{C}_0(X)$ , the  $C^*$  algebra of continuous functions vanishing at  $\infty$ ; however, for a Polish metric space, to be locally compact is known to be properly  $\Pi_1^1$ .)

Camerlo and Gao [8] proved that graph isomorphism is Borel reducible to homeomorphism of totally disconnected compact metric spaces (i.e., separable Stone spaces). One notes that countable compact metric spaces  $X$  won't

work here, because  $X$  is scattered and hence given by the Cantor-Bendixson rank  $\alpha$ , together with the size of the last set  $X^{(\alpha)}$ .

The main question remains open.

**Question 20.1.** *Determine the complexity with respect to  $\leq_B$  of  $\cong_h$  for compact metric spaces.*

In contrast, in the computable case the complexity is known to be as large as possible.

**Theorem 20.2.** *Homeomorphism of compact computable metric spaces is complete for  $\Sigma_1^1$  equivalence relations on  $\omega$  with respect to computable reductions.*

*Proof.* Friedman et al. [15] showed this for isomorphism of computable graphs. It can be verified that the construction Camerlo and Gao [8] use for providing their Borel reduction is effective. Hence, if the given graph is computable, then uniformly in its index they build a compact computable metric space.  $\square$

## 21. THE COMPLEXITY OF PARTICULAR ISOMETRIES

Let us return to the leading questions posed initially. It appears that Questions (b) and (c) are closely connected:

*It is easy to detect that  $X$  is similar to  $Y \Leftrightarrow$*

*we can determine from  $X, Y$  a means via which the similarity holds.*

We will provide some evidence for this thesis, first for compact metric spaces, and then for metric measure spaces studied by Gromov [23] and Vershik. For a function  $g$ , let  $g'$  be the halting problem relative to the graph of  $g$ .

**Theorem 21.1** (Melnikov, Nies [28]). *Let  $X, Y$  be compact metric spaces. Let  $A$  be an oracle Turing equivalent to the Turing jump of (the presentation of)  $X$  together with  $Y$ .*

- (a) *If  $X \cong_i Y$  then there is an isometry  $g$  such that  $g' \leq_T A''$ .*
- (b) *there are isometric compact computable metric spaces  $X, Y$  with no isometry  $g \leq_T \emptyset'$ .*

For (a) note that it suffices to build  $g$  an isometric embedding. By compactness we can view embeddings as branches on a subtree  $T \subseteq \omega^{<\omega}$  with an  $A'$  computable bound on the level size. Now apply the low basis theorem relative to  $A'$  in order to obtain  $g$ .

A metric measure (m.m.) space has the form  $T = (X, \mu, d)$  where  $(X, d)$  is Polish,  $\mu$  a Borel probability measure. We may assume that  $\mu U > 0$  for any non-empty open  $U$ ; otherwise, replace  $X$  by the least conull closed set.

**Theorem 21.2** (Gromov (1997), see [23]). *Measure-preserving isometry of m.m. spaces is smooth.*

Gromov used as invariants the sequence of distributions  $D_n$  of the distance matrix of  $n$  randomly chosen points. He used moments to show that  $\langle D_n \rangle_{n \in \mathbb{N}}$  is a complete invariant for the m.m. space  $T$ . Note that in the subsequence lemma, there is a typo. It should say  $\liminf$  there.

In 1996 Anatoly Vershik [39] gave a proof as well; also see the survey [40]. He describes  $T$  by the single invariant  $D_T$ , the distribution of the distance matrix of a randomly chosen infinite sequence  $(x_i)$ . More formally,  $D_T$  is the push forward measure of  $d(x_i, x_k)$  on the space  $\mathcal{M} \subseteq \mathbb{R}^{\omega \times \omega}$  of distance matrices. He used a form of the law of large numbers to reconstruct  $T$  from  $D_T$ . The 2006 paper by Cameron and Vershik is also relevant here.

We can give an effective analysis of Vershik's proof. Let  $\mathcal{O} \subseteq \omega$  be some  $\Pi_1^1$  complete set.

**Theorem 21.3.** *Suppose  $T_1, T_2$  are computable m.m. spaces (that is, the measure of Boolean combinations of open balls is uniformly computable). Then there is a measure-preserving isometry  $\Theta$  such that  $\Theta \leq_T \mathcal{O}$ .*

*Proof.* Recall that  $\mathcal{M}$  is the Polish space of distance matrices. Following Vershik, we have canonical maps  $F_k: T_k^\omega \rightarrow \mathcal{M}$ ,  $\bar{x} \rightarrow \langle d(x_i, x_k) \rangle_{i,k \in \mathbb{N}}$ . Let  $D_k = F_k \mu_k^\omega$  be the push forward measure on  $\mathcal{M}$ . This is the distribution of the distance matrix for a randomly picked sequence of points in  $T_k$ .  $\square$

The main source of complexity is that one has to pick an element  $r$  in a non-empty  $\Sigma_1^1$  class of distance matrices. By Gandy basis theorem, there is such an  $r$  with  $\mathcal{O}^r \leq_h \mathcal{O}$ . Ongoing work of Melnikov and Nies reduces this complexity to  $\Delta_3^0$ .

## Part 5. Other topics

### 22. YU: A NOTE ON THE GREENBERG-MONTALBAN-SLAMAN THEOREM

Greenberg, Montalban and Slaman prove the following theorem.

**Theorem 22.1** (Greenberg, Montalban and Slaman [22]). *Assume that  $\omega_1$  is inaccessible in  $L$ . For any countable structure  $\mathcal{M}$ , if the set  $A = \{x \mid \exists \mathcal{N} \in L[x](\mathcal{N} \cong \mathcal{M})\}$  contains all the nonconstructible reals, then  $A = 2^\omega$ .*

We prove that, under the weaker assumption that  $\omega_1^L < \omega_1$ , Theorem 22.1 remains true for any  $\Sigma_2^1$ -equivalence relation.

*Proof.* For any equivalence relation  $E$ , reduction  $\leq_r$  over  $2^\omega$  and real  $x \in 2^\omega$ , let

$$\text{Spec}_{E,r}(x) = \{y \mid \exists z \leq_r y(E(z, x))\}$$

be the  $(E, r)$ -spectrum of  $x$ .

Let  $E$  be a  $\Sigma_2^1$ -relation and  $x$  be a real so that  $\text{Spec}_{E,L}(x) \supseteq \{z \mid z \notin L\}$ . Since  $E$  is  $\Sigma_2^1$ , there must be some  $\Pi_1^1$ -relation  $R_0 \subseteq (2^\omega)^3$  so that

$$\forall y \forall z (E(y, z) \leftrightarrow \exists s R_0(y, z, s)).$$

By the Shoenfield absoluteness,

$$\forall y \forall z (E(y, z) \leftrightarrow \exists s \in L_{\omega_1^{L[y \oplus z]}}[y \oplus z] R_0(y, z, s)).$$

In particular,

$$\forall y (E(y, x) \leftrightarrow \exists s \in L_{\omega_1^{L[y \oplus x]}}[y \oplus x] R_0(y, x, s)).$$

Note that, by the assumption, the set  $\text{Spec}_{E,L}(x)$  is  $\Sigma_2^1(x)$  and conull.

Since  $\omega_1^L < \omega_1$ , there are conull many  $L$ -random reals. So the set  $\{y \mid \omega_1^L = \omega_1^{L[y]}\}$  is conull. We may also assume that  $\omega_1^{L[x]} = \omega_1^L$ . Then the set  $\{y \mid \omega_1^{L[x \oplus y]} = \omega_1^L\}$  is also conull.

Then  $z \in \text{Spec}_{E,L}(x) \leftrightarrow$

$$\exists t \exists y \exists s (t \text{ codes a well ordering} \wedge y \in L_{|t|}[z] \wedge s \in L_{|t|}[y \oplus x] \wedge R_0(y, x, s)).$$

For any real  $t$  coding a well ordering, let

$$z \in R_{1,t} \leftrightarrow \exists y \in L_{|t|}[z] \exists s \in L_{|t|}[y \oplus x] (R_0(y, x, s)).$$

Then  $R_{1,t} \subseteq \text{Spec}_{E,L}(x)$  is a  $\Pi_1^1(t \oplus x)$ -set and so measurable. Moreover, if  $z$  is  $L[x]$ -random, then  $z \in \text{Spec}_{E,L}(x)$  if and only if  $z \in R_{1,t}$  for some real  $t \in L$  coding a well ordering. Since  $\mu(\text{Spec}_{E,L}(x)) = 1$  and  $\omega_1^L < \omega_1$ , there must be some  $t \in L$  coding a well ordering so that  $\mu(R_{1,t}) > 0$ . Fix such a real  $t_0 \in L$ . Then there must be some formula  $\varphi$  in the set theory language so that the set

$$R_{1,t_0,\varphi} = \{z \mid \exists y \exists s \in L_{|t_0|}[y \oplus x] (\forall n (n \in y \leftrightarrow L_{|t_0|}[z] \models \varphi(n)) \wedge R_0(y, x, s))\}$$

has positive measure. Then there must be some  $\sigma \in 2^{<\omega}$  so that

$$\mu(R_{1,t_0,\varphi} \cap [\sigma]) > \frac{7}{8} \cdot 2^{-|\sigma|}.$$

Were  $x$  constructible, then  $R_{1,t_0} \cap [\sigma]$  would contain a constructible real. Now we try to get rid of the parameter  $x$ .

Let

$$S = \{r \mid \mu(\{z \succ \sigma \mid \exists y \exists s \in L_{|t_0|}[y \oplus r] (\forall n (n \in y \leftrightarrow L_{|t_0|}[z] \models \varphi(n)) \wedge R_0(y, r, s))\}) > \frac{3}{4} \cdot 2^{-|\sigma|}\}.$$

Then  $S$  is a  $\Pi_1^1(t_0)$ -set and every real in  $S$  is  $E$ -equivalent to  $x$ . Since  $x \in S$ , we have that  $S$  is not empty. Thus there must be some  $t_0$ -constructible, and so constructible, real in  $S$ .

This completes the proof.  $\square$

By a similar method, one also can prove:

**Proposition 22.2.** *For any  $\Pi_1^1$ -equivalence relation  $E$  and real  $x$ , if  $\text{Spec}_{E,h}(x) \supseteq \{z \mid z \notin \Delta_1^1\}$ , then  $\text{Spec}_{E,h}(x) = 2^\omega$ .*

Let  $MA$  be Martin's axiom, By a similar method, one also can prove

**Theorem 22.3.** *Assume that  $MA \wedge 2^{\aleph_0} > \aleph_1 + \omega_1^L = \omega_1$ . Then for any real  $x_0$ , and  $\Sigma_2^1(x_0)$ -relation  $E$ , if  $\text{Spec}_{E,L}(x) \supseteq \{z \in 2^\omega \mid z \notin L[x_0]\}$ , then  $\text{Spec}_{E,L}(x) = 2^\omega$ .*

So the large cardinal assumption in [22] is unnecessary.

Note: *Normality of a real relative to non-integral bases, and uniform distribution* has moved to the 2014 Blog.

### 23. TURETSKY: $K^X \geq_T X$

Proved by Miller and Turetsky, and then vastly simplified by Bienvenu. Let  $K$  denote prefix free descriptive string complexity.

**Proposition 23.1.** *For any real  $X$ ,  $K^X \geq_T X$ .*

*Proof.*  $X$  is  $X$ -trivial. That is,  $K^X(X \upharpoonright_n) \leq K^X(n) + c$ . Note that  $K^X$  can compute the tree  $\{\sigma \in 2^{<\omega} : K^X(\sigma) \leq K^X(|\sigma|) + c\}$ . This tree has finitely many infinite paths, and  $X$  is one of them. As an isolated path,  $K^X$  can compute  $X$ .  $\square$

### 24. NIES: NOTES ON A THEOREM OF HIRSCHFELDT, KUYPER AND SCHUPP REGARDING COARSE COMPUTATION AND $K$ -TRIVIALITY

Recall that we write  $X \leq_{ibT} Y$  if  $X \leq_T Y$  with use function bounded by the identity. When building prefix free machines, we use the terminology of [30, Section 2.3] such as Machine Existence Theorem (also called the Kraft-Chaitin Theorem), bounded request set etc.

Hirschfeldt, Kuiper and Schupp (2013) proved the following in slightly different language.

**Theorem 24.1.** *Let  $Y$  be a  $\Delta_2^0$  set of positive effective Hausdorff dimension. There is a cost function  $\mathbf{c}$  such that  $A \models \mathbf{c}$  implies  $A \leq_{ibT} D$  for any set  $D$  with  $\bar{\rho}(D \Delta Y) = 0$ .*

*Moreover, if  $Y$  is  $\omega$ -c.e., then  $\mathbf{c}$  can be chosen to be benign.*

*Proof.* The proof given here extends a similar result in [20], and also [31, Thm 5.5].

By the hypothesis on  $Y$  there is a positive rational  $\delta$  such that

$$3\delta < \liminf_n K(Y \upharpoonright_n)/n.$$

Let  $\langle Y_s \rangle$  be a computable approximation of  $Y$ . To help with building a reduction procedure for  $A \leq_{ibT} D$ , via the Machine Existence Theorem we give prefix-free descriptions of initial segments  $Y_s \upharpoonright_e$ . On input  $x$ , if at a stage  $s > x$ ,  $e$  is least such that  $Y(e)$  has changed between stages  $x$  and  $s$ , then we still hope that  $Y_s \upharpoonright_e$  is the final version of  $Y \upharpoonright_e$ . So whenever  $A(x)$  changes at such a stage  $s$ , we give a description of  $Y_s \upharpoonright_e$  of length  $\lfloor \delta e \rfloor$ . We will define an appropriate cost function  $\mathbf{c}$  so that a set  $A$  that obeys  $\mathbf{c}$  changes little enough that we can provide all the descriptions needed.

To ensure that  $A \leq_{ibT} D$ , we define a computation  $\Gamma(D \upharpoonright_x)$  with output  $A(x)$  at the least stage  $t \geq x$  such that  $Y_t \Delta D \upharpoonright_e$  has sufficiently few 1's for each  $e \leq x$ . Then  $A(x)$  cannot change at any stage  $s > t$  (for almost all  $x$ ), for otherwise  $Y_s \upharpoonright_e$  would receive a description of length  $\lfloor \delta e \rfloor$ , where  $e$  is least such that  $Y(e)$  has changed between  $x$  and  $s$ .

We give the details. Let  $H$  denote the binary Bernoulli entropy. Choose a rational  $\beta > 0$  such that  $H(\beta) \leq \delta$ . This implies that no more than  $2^{\delta n}$

strings  $v$  of length  $n$  have at most  $\beta n$  many 1's (see Wikipedia page on binomial coefficients). Therefore, for each such string  $v$ , we have

$$(5) \quad K(v) \leq \delta n + 2 \log n + O(1).$$

Next we give a definition of a cost function  $\mathbf{c}$ . Let  $\mathbf{c}(x, s) = 0$  for each  $x \geq s$ . If  $x < s$ , and  $e < x$  is least such that  $Y_{s-1}(e) \neq Y_s(e)$ , let

$$(6) \quad \mathbf{c}(x, s) = \max(\mathbf{c}(x, s-1), 2^{-\lfloor \delta e \rfloor}).$$

We show that  $A \models \mathbf{c}$  implies  $A \leq_{ibT} D$  for any set  $A$ . We may suppose that  $\mathbf{c}(A_s) \leq 1$ . Enumerate a bounded request set  $L$  as follows. When  $A_{s-1}(x) \neq A_s(x)$  and  $e$  is least such that  $e = x$  or  $Y_{t-1}(e) \neq Y_t(e)$  for some  $t \in [x, s)$ , put the request  $\langle \lfloor \delta e \rfloor + 1, Y_s \upharpoonright_e \rangle$  into  $L$ . Then  $L$  is indeed a bounded request set.

We show  $A \leq_{ibT} D$ . Choose  $e_0$  with  $4 \log(e_0) \leq \delta e$  and for each  $e \geq e_0$  the number of 1's in  $(Y \triangle D) \upharpoonright_e$  is at most  $\beta e/2$ . Choose  $s_0 \geq e_0$  such that  $Y_s \upharpoonright_{e_0}$  is stable for each  $s \geq s_0$ .

Given an input  $x \geq s_0$ , using  $D$  as an oracle, compute  $t > x$  such that

$$\forall e. e_0 \leq e \leq x[(Y_t \triangle D) \upharpoonright_x \text{ has at most } \beta e/2 \text{ many 1's}].$$

We claim that  $A(x) = A_t(x)$ . Otherwise  $A_s(x) \neq A_{s-1}(x)$  for some  $s > t$ . Let  $e \leq x$  be the largest number such that  $Y_r \upharpoonright_e = Y_t \upharpoonright_e$  for all  $r, t < r \leq s$ . If  $e < x$  then  $Y(e)$  changes in the interval  $(t, s]$  of stages. Hence, by the choice of  $t \geq s_0$ , we cause  $K(Y_s \upharpoonright_e) < \lfloor \delta e \rfloor + O(1)$ . Since  $e \geq e_0$ , the string  $(Y_s \triangle Y) \upharpoonright_e$  has at most  $\lfloor \beta e \rfloor$  many 1's. Thus, by (5),

$$K(Y \upharpoonright_e) \leq K(Y_s \upharpoonright_e) + K((Y_s \triangle Y) \upharpoonright_e) + O(1) \leq \delta e + 4 \log e + O(1).$$

This contradicts the definition of  $\delta$  for  $x$  large enough.  $\square$

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